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Theorem on Fixed Points in Fuzzy 3-Metric Space

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Abstract

Our study examines fixed point theory within the context of fuzzy 3-metric spaces, which are a logical extension of traditional fuzzy metric spaces that take three-variable relationships into account. In contrast to the conventional Banach-type or Ćirić-type conditions that have been previously examined in the literature, our main contribution is the introduction of a novel generalized contractive condition.In this new contractive framework, we prove the existence and uniqueness of fixed points on fuzzy 3-metric spaces, and construct a fixed point theorem. The scope of fixed point results in generalized fuzzy settings is expanded by this development, which also makes them more applicable to systems with complex interactions and multi-way uncertainty. To illustrate the usefulness of our result, examples are provided.

Keywords: Fixed point, Fuzzy 2-metric space, Fuzzy 3-metric space.

1 | Introduction

We are aware that there are two categories of fixed points that are under discussion [1]. The first category, known as Banach fixed point theorems, deals with contraction [2]. The second type is more complex and deals with compact mappings [3]. A crucial role is played by the metric fixed point theorem. In a variety of spaces, including Banach, Hilbert, cone, soft, and others, numerous writers have demonstrated the fixed point theorem. Zadeh first proposed the idea of fuzzy sets. Kaleva and Seikkala [4], Deng [5], Eklund and Gahler [6], and Kramosil and Michalek [7] have presented the idea of many approaches to fuzzy metric spaces. The fixed point theory has also been examined by numerous writers in these fuzzy metric spaces and for fuzzy mappings and numerous other individuals recently initiated studies on probabilistic 2-metric spaces.

Given the abstract qualities implied by the area function in Euclidean spaces, we know that 2-metric space is a real valued function of a point triple on a set X. Now, the volume function suggests 3-metric space, which

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is what is expected. Naturally, this is introduced in a different way than 2-metric space theory. A common fixed point theorem for three mappings in fuzzy metric space is proved in this study. The fuzzy 2-metric and fuzzy 3-metric spaces are also covered by this solution [8–10].

2 | Preliminaries

Definition 1. An operation in binary *: [0, 1] × [0, 1] → [0, 1] is referred to as a t-norm in ([0,1], *) if all of the following conditions are satisfied for every a, b, c, and d that are contained in [0,1]:

I. a*1 = a.

II. a*(b*c) = (a*b)*c.

III. a*b = b*a.

IV. a*b ≤ c*d whenever a ≤ c and b ≤ d.

Definition 2. If X is an arbitrary set, * is a continuous t-Norm, and M is a fuzzy set in $X^2 \times [0,\infty]$, and all x, y, and z ∈X satisfy the following conditions, then the 3-tuple (X, $\mathcal{M}$, *) is referred to as a fuzzy metric space (FM space).

FM-1 For all t >0 if and only if x= y, $\mathcal{M}(x,y,t) = 1$,

FM-2 $\mathcal{M}(x,y,t) = \mathcal{M}(y,x,t)$ (symmetry),

FM-3 $\mathcal{M}(x,y,t) * \mathcal{M}(y,z,s) \leq \mathcal{M}(x,z,t+s)$,

FM-4 $\mathcal{M}(x,y,0) = 0$,

FM-5 $\lim_{t\to\infty} \mathcal{M}(x,y,t) = 1$,

FM-6 $\mathcal{M}(x,y,.): [0,1] \to [0,1]$

is left continuous,

Definition 3. Let (X, $\mathcal{M}$, *) is a fuzzy metric space.

I. A sequence $\{x_n\}$ in X can be represented by $\lim_{n\to\infty} x_n = x$ and is said to converge to a point x∈X. If $\lim_{n\to\infty} \mathcal{M}(x_n, x, t) = 1$, for all t > 0.

II. If for all t > 0 and p > 0, then $\lim_{n\to\infty} \mathcal{M}(x_{n+p}, x_n, t) = 1$ is a Cauchy sequence -a sequence $\{x_n\}$ in X.

III. A fuzzy metric space is complete if all of its Cauchy sequences converge.

Definition 4. If X is an arbitrary set, * is a continuous t-Norm, and $\mathcal{M}$ is a fuzzy set in $X^3 \times [0,\infty]$, and all x, y, z, and u∈ X satisfy the following conditions for all t_1, t_2, and t_3> 0, then the 3-tuple (X, $\mathcal{M}$, *) is referred to as a fuzzy 2-metric space.

FM'-1 For every t >0 if and only if x= y, $\mathcal{M}(x,y,z,t) = 1$,

FM'-2 $\mathcal{M}(x,y,z,t) = \mathcal{M}(x,z,y,t) = \mathcal{M}(y,z,x,t)$ (about three variables symmetric),

FM'-3 $\mathcal{M}(x,y,u,t_1) * \mathcal{M}(x,u,z,t_2) * \mathcal{M}(x,u,z,t_3) \leq \mathcal{M}(x,y,z,t_1+t_2+t_3)$,(Tetrahedron inequality in 2-metric space is equivalent to this).

The probability that the area of the triangle is less than t can be deduced from the function value $\mathcal{M}(x,y,z,t)$,

FM'-4 $\mathcal{M}(x,y,z,0) = 0$,

FM'-5 $\mathcal{M}(x,y,z,.): [0,1) \to [0,1]$ is left continuous.

Definition 5. Assume that (X, $\mathcal{M}$, *) is a fuzzy 2-metric space:

If $\lim_{n\to\infty} \mathcal{M}(x_n, a, t) = 1$, for any a in X and t > 0, then a sequence $\{x_n\}$ in fuzzy 2 metric space X is said to converge to a point $x \in X$.

If $\lim_{n\to\infty} \mathcal{M}(x_{n+p}, x_n, a, t) = 1$, for any a in X and t> 0, p > 0, then a sequence $\{x_n\}$ in fuzzy 2-metric space X is referred to as a Cauchy sequence.

Every Cauchy sequence that converges in a fuzzy two-metric space is considered complete.

Definition 6. If X is an arbitrary set, * is a continuous t-Norm, and $\mathcal{M}$ is a fuzzy set in $X^4 \times [0,\infty]$, and all x, y, z, u, and w ∈ X and t_1, t_2, t_3 and t_4> 0, then the 3-tuple (X, $\mathcal{M}$, *) is referred to as a fuzzy 3-metric space.

FM”-1 For every t > 0 if and only if x= y, $\mathcal{M}(x,y,z,w,t) = 1$.

FM”-2 $\mathcal{M}(x,y,z,w,t) = \mathcal{M}(x,w,z,y,t) = \mathcal{M}(y,z,w,x,t) = \mathcal{M}(z,w,x,y,t) = \cdots$ (about three variables symmetric).

FM”-3 $\mathcal{M}(x,y,z,u,t_1) * \mathcal{M}(x,y,u,w,t_2) * \mathcal{M}(x,u,z,w,t_3) * \mathcal{M}(u,y,z,w,t_4) \leq \mathcal{M}(x,y,z,w,t_1+t_2+t_3+t_4)$.

FM”-4 $\mathcal{M}(x,y,z,w,0) = 0$.

FM”-5 $\mathcal{M}(x,y,z,w,.): [0,1) \to [0,1]$ is left continuous.

Definition 7. Assume that (X, $\mathcal{M}$, *) is a fuzzy 3-metric space:

If $\lim_{n\to\infty} \mathcal{M}(x_n, x, a, b, t) = 1$, for any a,b in X and t > 0, then a sequence $\{x_n\}$ in fuzzy 3 metric space X is said to converge to a point $x \in X$.

If $\lim_{n\to\infty} \mathcal{M}(x_{n+p}, x_n, a, b, t) = 1$, for any a,b in X and t> 0, p > 0, then a sequence $\{x_n\}$ in fuzzy 3-metric space X is referred to as a Cauchy sequence.

Every Cauchy sequence that converges in a fuzzy 3-metric space is considered complete.

3|Main Results

Theorem 1. Let (X, $\mathcal{M}$, *) be a fuzzy metric space that is complete and has the condition (FM-5). If S & $\mathcal{T}$ are continuous mapping of X in X if there is a continuous mapping A of X into $\mathcal{S}$(X)⋂$\mathcal{T}$(X) that commutes with $\mathcal{S}$ & $\mathcal{T}$, then $\mathcal{S}$ and $\mathcal{T}$ have a common fixed point in X and $\mathcal{M}(Ax, Ay, qt) \geq \frac{\min\{\mathcal{M}(\mathcal{S}x,Ay,t),\mathcal{M}(\mathcal{T}x,Ax,t),\mathcal{M}(\mathcal{T}y,Ax,t)\}}{\min\{\mathcal{M}(\mathcal{S}x,Ay,t),\mathcal{M}(\mathcal{T}y,Ax,t)\}}$ for all x,y,z ∈X, t>0 and 0<q<1. A, $\mathcal{T}$, and $\mathcal{S}$ then have a unique common fixed point.

Proof: We will demonstrate that $\{Ax_n\}$ is a Cauchy sequence by defining sequences $\{x_n\}$ such that $Ax_{2n} = \mathcal{S}x_{2n-1}$ and $Ax_{2n-1} = \mathcal{S}\mathcal{T}x_{2n}$, where $n = 1,2,,3,\ldots$, In order to do this, enter $x = x_{2n}$ and $y = x_{2n+1}$. Then, we write

$$\mathcal{M}(Ax_{2n}, Ax_{2n+1}, qt) \geq \frac{\min\{\mathcal{M}(\mathcal{S}x_{2n},Ax_{2n+1},t),\mathcal{M}(\mathcal{T}x_{2n},Ax_{2n},t),\mathcal{M}(\mathcal{T}x_{2n+1},Ax_{2n},t)\}}{\min\{\mathcal{M}(\mathcal{S}x_{2n},Ax_{2n+1},t),\mathcal{M}(\mathcal{T}x_{2n+1},Ax_{2n},t)\}}$$

$$\geq \frac{\min\{\mathcal{M}(Ax_{2n+1},Ax_{2n+1},t),\mathcal{M}(Ax_{2n-1},Ax_{2n},t),\mathcal{M}(Ax_{2n},Ax_{2n},t)\}}{\min\{\mathcal{M}(Ax_{2n+1},Ax_{2n+1},t),\mathcal{M}(Ax_{2n},Ax_{2n},t)\}} \geq \mathcal{M}(Ax_{2n-1}, Ax_{2n}, t).$$

[since, $\mathcal{M}(Ax_{2n+1}, Ax_{2n+1}, t) = \mathcal{M}(Ax_{2n}, Ax_{2n}, t) = 1] \geq \mathcal{M}\left(Ax_{2n-1}, Ax_{2n}, \frac{t}{q}\right)$.

Consequently, $\mathcal{M}(Ax_{2n}, Ax_{2n+1}, qt) \geq \mathcal{M}\left(Ax_{2n-1}, Ax_{2n}, \frac{t}{q}\right)$

Using induction $\mathcal{M}(Ax_{2k}, Ax_{2m+1}, qt) \geq \mathcal{M}\left(Ax_{2k-1}, Ax_{2m}, \frac{t}{q}\right)$

Further, for each k and m in N, if $2m + 1 > 2k$ then

$$\mathcal{M}(Ax_{2k}, Ax_{2m+1}, qt) \geq M\left(Ax_{2k-1}, Ax_{2m}, \frac{t}{q}\right) \dots \geq \mathcal{M}\left(Ax_0, Ax_{2m+1-2k}, \frac{t}{q^{2k}}\right).$$

If $2k > 2m + 1$ then

$$\mathcal{M}(Ax_{2k}, Ax_{2m+1}, qt) \geq \mathcal{M}\left(Ax_{2k-1}, Ax_{2m}, \frac{t}{q}\right) \dots \geq \mathcal{M}\left(Ax_{2k-(2m+1)}, Ax_0, \frac{t}{q^{2m+1}}\right),$$

Simple induction yields

$$\mathcal{M}(Ax_n, Ax_{n+p}, qt) \geq \mathcal{M}\left(Ax_0, Ax_p, \frac{t}{q^n}\right),$$

in the case of $n = 2k, p = 2m + 1$ and by (FM-3)

$$\mathcal{M}(Ax_n, Ax_{n+p}, qt) \geq \mathcal{M}\left(Ax_0, Ax_1, \frac{t}{2q^n}\right) * \mathcal{M}\left(Ax_0, Ax_p, \frac{t}{2q^n}\right),$$

for any positive integer p and n in N, if$n = 2k, p = 2m$ or $n = 2k + 1, p = 2m$, By observing that $\mathcal{M}\left(Ax_0, Ax_p, \frac{t}{2q^n}\right) \to 1$ as $n \to \infty$ for each positive integer p & n in N.

The sequence $\{Ax_n\}$ is consequently Cauchy. There are $z = \lim_{n\to\infty} Ax_n$ and $z = \lim_{n\to\infty} \mathcal{S}x_{2n-1} = \lim_{n\to\infty} \mathcal{T}x_{2n}$ since the space X is complete. As a result, $Az = \mathcal{S}z = \mathcal{T}z$ and

$$\begin{aligned}
&\mathcal{M}(Az, A^2z, qt) \geq \mathcal{M}(Az, AAz, qt) \geq \\
&\min\{\mathcal{M}(\mathcal{S}z, AAz, t), \mathcal{M}(\mathcal{T}z, Az, t), \mathcal{M}(\mathcal{T}Az, Az, t)\} \geq \\
&\min\{\mathcal{M}(\mathcal{S}z, A\mathcal{T}z, t), \mathcal{M}(Az, Az, t), \mathcal{M}(A\mathcal{T}z, Az, t)\} \geq \\
&\min\{\mathcal{M}(\mathcal{S}z, A\mathcal{T}z, t), \mathcal{M}(Az, Az, t), \mathcal{M}(A\mathcal{T}z, \mathcal{S}z, t)\} \geq \mathcal{M}(\mathcal{S}z, A\mathcal{T}z, t) \geq \\
&\mathcal{M}(\mathcal{S}z, AAz, t) \geq \mathcal{M}(Az, A^2z, t) \dots \geq \mathcal{M}\left(Az, A^2z, \frac{t}{q^n}\right).
\end{aligned}$$

Given that $\lim_{n\to\infty} \mathcal{M}\left(Az, A^2z, \frac{t}{2q^n}\right) = 1$ Consequently, $Az = A^2z$

Z is therefore the common fixed point of A, $\mathcal{S}$, and $\mathcal{T}$. Let w ($w \neq z$) be another common fixed point of $\mathcal{S}$, $\mathcal{T}$, and A for uniqueness. Then, we write

$$\mathcal{M}(Az, Aw, qt) \geq \frac{\min\{\mathcal{M}(\mathcal{S}z, Aw, t), \mathcal{M}(\mathcal{T}z, Az, t), \mathcal{M}(\mathcal{T}w, Az, t)\}}{\min\{\mathcal{M}(\mathcal{S}z, Aw, t), \mathcal{M}(\mathcal{T}w, Az, t)\}}.$$

This indicates $\mathcal{M}(z, w, qt) \geq \mathcal{M}(z, w, t)$.

Thus, we write $z = w$ using the lemma. The proof of *Theorem 1* is now complete. Theorem for fuzzy 2-metric spaces is now proved.

Theorem 2. Let (X, $\mathcal{M}$, *) be a fuzzy 2-metric space that is complete and has the condition. If S & $\mathcal{T}$ are continuous mapping of X in X if there is a continuous mapping A of X into $\mathcal{S}(X) \cap \mathcal{T}(X)$ that commutes with $\mathcal{S}$ & $\mathcal{T}$, then $\mathcal{S}$ and $\mathcal{T}$ have a common fixed point in X and $\mathcal{M}(Ax, Ay, a, qt) \geq \frac{\min\{\mathcal{M}(\mathcal{S}x, Ay, a, t), \mathcal{M}(\mathcal{T}x, Ax, a, t), \mathcal{M}(\mathcal{T}y, Ax, a, t)\}}{\min\{\mathcal{M}(\mathcal{S}x, Ay, a, t), \mathcal{M}(\mathcal{T}y, Ax, a, t)\}}$ forall $x, y, a, z \in X$, $t>0$ and $0<q<1$. $\lim_{t\to\infty} \mathcal{M}(x, y, z, t) = 1$ for all $x, y, z \in X$. A, $\mathcal{T}$, and $\mathcal{S}$ then have a unique common fixed point.

Proof: We will demonstrate that $\{Ax_n\}$ is a Cauchy sequence by defining sequences $\{x_n\}$ such that $Ax_{2n} = \mathcal{S}x_{2n-1}$ and $Ax_{2n-1} = \mathcal{S}\mathcal{T}x_{2n}$, where $n = 1,2,,3, \dots$, In order to do this, enter $x = x_{2n}$ and $y = x_{2n+1}$. Then, we write

$$\mathcal{M}(Ax_{2n}, Ax_{2n+1}, a, qt) \geq \frac{\min\{\mathcal{M}(\mathcal{S}x_{2n}, Ax_{2n+1}, a, t), \mathcal{M}(\mathcal{T}x_{2n}, Ax_{2n}, a, t), \mathcal{M}(\mathcal{T}x_{2n+1}, Ax_{2n}, a, t)\}}{\min\{\mathcal{M}(\mathcal{S}x_{2n}, Ax_{2n+1}, a, t), \mathcal{M}(\mathcal{T}x_{2n+1}, Ax_{2n}, a, t)\}}$$

$\geq \frac{\min\{\mathcal{M}(Ax_{2n+1},Ax_{2n+1},a,t),\mathcal{M}(Ax_{2n-1},Ax_{2n},a,t),\mathcal{M}(Ax_{2n},Ax_{2n},a,t)\}}{\min\{\mathcal{M}(Ax_{2n+1},Ax_{2n+1},a,t),\mathcal{M}(Ax_{2n},Ax_{2n},a,t)\}} \geq \mathcal{M}(Ax_{2n-1}, Ax_{2n}, a, t)$

[since, $\mathcal{M}(Ax_{2n+1}, Ax_{2n+1}, a, t) = \mathcal{M}(Ax_{2n}, Ax_{2n}, a, t) = 1] \geq \mathcal{M}\left(Ax_{2n-1}, Ax_{2n}, a, \frac{t}{q}\right)$.

Consequently, $\mathcal{M}(Ax_{2n}, Ax_{2n+1}, a, qt) \geq \mathcal{M}\left(Ax_{2n-1}, Ax_{2n}, a, \frac{t}{q}\right)$.

Using induction $\mathcal{M}(Ax_{2k}, Ax_{2m+1}, a, qt) \geq \mathcal{M}\left(Ax_{2k-1}, Ax_{2m}, a, \frac{t}{q}\right)$.

Further, for each k and m in N, if $2m + 1 > 2k$ then

$\mathcal{M}(Ax_{2k}, Ax_{2m+1}, a, qt) \geq \mathcal{M}\left(Ax_{2k-1}, Ax_{2m}, a, \frac{t}{q}\right) \dots \geq \mathcal{M}\left(Ax_0, Ax_{2m+1-2k}, \frac{t}{q^{2k}}\right)$

If $2k > 2m + 1$,

then

$\mathcal{M}(Ax_{2k}, Ax_{2m+1}, a, qt) \geq \mathcal{M}\left(Ax_{2k-1}, Ax_{2m}, a, \frac{t}{q}\right) \dots \geq \mathcal{M}\left(Ax_{2k-(2m+1)}, Ax_0, a, \frac{t}{q^{2m+1}}\right)$.

Simple induction yields $\mathcal{M}(Ax_n, Ax_{n+p}, a, qt) \geq \mathcal{M}\left(Ax_0, Ax_p, a, \frac{t}{q^n}\right)$.

In the case of $n = 2k, p = 2m + 1$ and by (FM-4).

$\mathcal{M}(Ax_n, Ax_{n+p}, a, qt) \geq \mathcal{M}\left(Ax_0, Ax_1, a, \frac{t}{3q^n}\right) * \mathcal{M}\left(Ax_0, Ax_p, a, \frac{t}{3q^n}\right) * \mathcal{M}\left(Ax_1, Ax_p, a, \frac{t}{3q^n}\right)$.

For any positive integer p & n in N, if $n = 2k, p = 2m$ or $n = 2k + 1, p = 2m$, By observing that $\mathcal{M}\left(Ax_0, Ax_p, a, \frac{t}{q^n}\right) \to 1$ as $n \to \infty$ for each positive integer p & n in N.

The sequence $\{Ax_n\}$ is consequently Cauchy. There are $z = \lim_{n\to\infty} Ax_n$ and $z = \lim_{n\to\infty} \mathcal{S}x_{2n-1} = \lim_{n\to\infty} \mathcal{T}x_{2n}$ since the space X is complete. As a result, $Az = \mathcal{S}z = \mathcal{T}z$ and

$\mathcal{M}(Az, A^2z, a, qt) \geq \mathcal{M}(Az, AAz, a, qt)$

$\geq \min\{\mathcal{M}(\mathcal{S}z, AAz, a, t), \mathcal{M}(\mathcal{T}z, Az, a, t), \mathcal{M}(\mathcal{T}Az, Az, a, t)\}$

$\geq \min\{\mathcal{M}(\mathcal{S}z, A\mathcal{T}z, a, t), \mathcal{M}(Az, Az, a, t), \mathcal{M}(A\mathcal{T}z, Az, a, t)\}$

$\geq \min\{\mathcal{M}(\mathcal{S}z, A\mathcal{T}z, a, t), \mathcal{M}(Az, Az, a, t), \mathcal{M}(A\mathcal{T}z, \mathcal{S}z, a, t)\}$

$\geq \mathcal{M}(\mathcal{S}z, A\mathcal{T}z, a, t) \geq \mathcal{M}(\mathcal{S}z, AAz, a, t) \geq \mathcal{M}(Az, A^2z, a, t) \dots \geq \mathcal{M}\left(Az, A^2z, a, \frac{t}{q^n}\right)$.

Given that $\lim_{n\to\infty} \mathcal{M}\left(Az, A^2z, a, \frac{t}{2q^n}\right) = 1$ Consequently, $Az = A^2z$.

Z is therefore the common fixed point of A, $\mathcal{S}$, and $\mathcal{T}$.

Let w ($w \neq z$) be another common fixed point of $\mathcal{S}$, $\mathcal{T}$, and A for uniqueness. Then, using (5), we write

$$\mathcal{M}(Az, Aw, a, qt) \geq \frac{\min\{\mathcal{M}(\mathcal{S}z, Aw, a, t), \mathcal{M}(\mathcal{T}z, Az, a, t), \mathcal{M}(\mathcal{T}w, Az, a, t)\}}{\min\{\mathcal{M}(\mathcal{S}z, Aw, a, t), \mathcal{M}(\mathcal{T}w, Az, a, t)\}}.$$

This indicates $\mathcal{M}(z, w, a, qt) \geq \mathcal{M}(z, w, a, t)$.

Thus, we write $z = w$ using the lemma. The proof of *Theorem 2* is now complete. Theorem for fuzzy 3-metric spaces is now proved.

Theorem 3. Let (X, $\mathcal{M}$, *) be a fuzzy 3-metric space that is complete and has the condition . If S & $\mathcal{T}$ are continuous mapping of X in X if there is a continuous mapping A of X into $\mathcal{S}$(X)$\cap\mathcal{T}$(X) that commutes with $\mathcal{S}$ & $\mathcal{T}$, then $\mathcal{S}$ and $\mathcal{T}$ have a common fixed point in X and $\mathcal{M}(\mathrm{Ax},\mathrm{Ay},\mathrm{a},\mathrm{b},\mathrm{qt}) \geq \frac{\min\{\mathcal{M}(\mathcal{S}\mathrm{x},\mathrm{Ay},\mathrm{a},\mathrm{b},\mathrm{t}),\mathcal{M}(\mathcal{T}\mathrm{x},\mathrm{Ax},\mathrm{a},\mathrm{b},\mathrm{t}),\mathcal{M}(\mathcal{T}\mathrm{y},\mathrm{Ax},\mathrm{a},\mathrm{b},\mathrm{t})\}}{\min\{\mathcal{M}(\mathcal{S}\mathrm{x},\mathrm{Ay},\mathrm{a},\mathrm{b},\mathrm{t}),\mathcal{M}(\mathcal{T}\mathrm{y},\mathrm{Ax},\mathrm{a},\mathrm{b},\mathrm{t})\}}$ For all x,y,a,b $\in X$, t>0 and 0<q<1. $\lim_{t\to\infty}\mathcal{M}(\mathrm{x},\mathrm{y},\mathrm{z},\mathrm{w},\mathrm{t}) = 1$ for all $\mathrm{x},\mathrm{y},\mathrm{z} \in \mathrm{X}$ A, $\mathcal{T}$, and $\mathcal{S}$ then have a unique common fixed point.

Proof: We will demonstrate that $\{\mathrm{Ax_n}\}$ is a Cauchy sequence by defining sequences $\{\mathrm{x_n}\}$ such that $\mathrm{Ax_{2n}} = \mathcal{S}\mathrm{x_{2n-1}}$ and $\mathrm{Ax_{2n-1}} = \mathcal{S}\mathcal{T}\mathrm{x_{2n}}$, where $\mathrm{n} = 1,2,,3,\dots.$, In order to do this, enter $\mathrm{x} = \mathrm{x_{2n}}$ and $\mathrm{y} = \mathrm{x_{2n+1}}$. Then, we write

$$\mathcal{M}(\mathrm{Ax_{2n}},\mathrm{Ax_{2n+1}},\mathrm{a},\mathrm{b},\mathrm{qt})$$
$$\geq \frac{\min\{\mathcal{M}(\mathcal{S}\mathrm{x_{2n}},\mathrm{Ax_{2n+1}},\mathrm{a},\mathrm{b},\mathrm{t}),\mathcal{M}(\mathcal{T}\mathrm{x_{2n}},\mathrm{Ax_{2n}},\mathrm{a},\mathrm{b},\mathrm{t}),\mathcal{M}(\mathrm{Tx_{2n+1}},\mathrm{Ax_{2n}},\mathrm{a},\mathrm{b},\mathrm{t})\}}{\min\{\mathcal{M}(\mathcal{S}\mathrm{x_{2n}},\mathrm{Ax_{2n+1}},\mathrm{a},\mathrm{b},\mathrm{t}),\mathcal{M}(\mathcal{T}\mathrm{x_{2n+1}},\mathrm{Ax_{2n}},\mathrm{a},\mathrm{b},\mathrm{t})\}}$$
$$\geq \frac{\min\{\mathcal{M}(\mathrm{Ax_{2n+1}},\mathrm{Ax_{2n+1}},\mathrm{a},\mathrm{b},\mathrm{t}),\mathcal{M}(\mathrm{Ax_{2n-1}},\mathrm{Ax_{2n}},\mathrm{a},\mathrm{b},\mathrm{t}),\mathcal{M}(\mathrm{Ax_{2n}},\mathrm{Ax_{2n}},\mathrm{a},\mathrm{b},\mathrm{t})\}}{\min\{\mathcal{M}(\mathrm{Ax_{2n+1}},\mathrm{Ax_{2n+1}},\mathrm{a},\mathrm{b},\mathrm{t}),\mathcal{M}(\mathrm{Ax_{2n}},\mathrm{Ax_{2n}},\mathrm{a},\mathrm{b},\mathrm{t})\}}$$
$$\geq \mathcal{M}(\mathrm{Ax_{2n-1}},\mathrm{Ax_{2n}},\mathrm{a},\mathrm{b},\mathrm{t})$$

[since, $\mathcal{M}(\mathrm{Ax_{2n+1}},\mathrm{Ax_{2n+1}},\mathrm{a},\mathrm{b},\mathrm{t}) = \mathcal{M}(\mathrm{Ax_{2n}},\mathrm{Ax_{2n}},\mathrm{a},\mathrm{b},\mathrm{t}) = 1] \geq$
$\mathcal{M}\left(\mathrm{Ax_{2n-1}},\mathrm{Ax_{2n}},\mathrm{a},\mathrm{b},\frac{\mathrm{t}}{\mathrm{q}}\right)$.

Consequently, $\mathcal{M}(\mathrm{Ax_{2n}},\mathrm{Ax_{2n+1}},\mathrm{a},\mathrm{qt}) \geq \mathcal{M}\left(\mathrm{Ax_{2n-1}},\mathrm{Ax_{2n}},\mathrm{a},\mathrm{b},\frac{\mathrm{t}}{\mathrm{q}}\right)$.

Using induction $\mathcal{M}(\mathrm{Ax_{2k}},\mathrm{Ax_{2m+1}},\mathrm{a},\mathrm{b},\mathrm{qt}) \geq \mathcal{M}\left(\mathrm{Ax_{2k-1}},\mathrm{Ax_{2m}},\mathrm{a},\mathrm{b},\frac{\mathrm{t}}{\mathrm{q}}\right)$.

Further, for each k and m in N, if $2\mathrm{m} + 1 > 2\mathrm{k}$ then

$$\mathcal{M}(\mathrm{Ax_{2k}},\mathrm{Ax_{2m+1}},\mathrm{a},\mathrm{b},\mathrm{qt}) \geq \mathcal{M}\left(\mathrm{Ax_{2k-1}},\mathrm{Ax_{2m}},\mathrm{a},\mathrm{b},\frac{\mathrm{t}}{\mathrm{q}}\right)\dots \geq$$
$$\mathcal{M}\left(\mathrm{Ax_0},\mathrm{Ax_{2m+1-2k}},\mathrm{a},\mathrm{b},\frac{\mathrm{t}}{\mathrm{q^{2k}}}\right) \text{ If } 2\mathrm{k} > 2\mathrm{m}+1,$$

then

$$\mathcal{M}(\mathrm{Ax_{2k}},\mathrm{Ax_{2m+1}},\mathrm{a},\mathrm{b},\mathrm{qt}) \geq \mathcal{M}\left(\mathrm{Ax_{2k-1}},\mathrm{Ax_{2m}},\mathrm{a},\mathrm{b},\frac{\mathrm{t}}{\mathrm{q}}\right)\dots \geq$$
$$\mathcal{M}\left(\mathrm{Ax_{2k-(2m+1)}},\mathrm{Ax_0},\mathrm{a},\mathrm{b},\frac{\mathrm{t}}{\mathrm{q^{2m+1}}}\right).$$

Simple induction yields

$$\mathcal{M}\left(\mathrm{Ax_n},\mathrm{Ax_{n+p}},\mathrm{a},\mathrm{b},\mathrm{qt}\right) \geq \mathcal{M}\left(\mathrm{Ax_0},\mathrm{Ax_p},\mathrm{a},\mathrm{b},\frac{\mathrm{t}}{\mathrm{q^n}}\right).$$

In the case of $\mathrm{n} = 2\mathrm{k}, \mathrm{p} = 2\mathrm{m}+1$ and by (FM-3)

$$\mathcal{M}\left(\mathrm{Ax_n},\mathrm{Ax_{n+p}},\mathrm{a},\mathrm{b},\mathrm{qt}\right) \geq \mathcal{M}\left(\mathrm{Ax_0},\mathrm{Ax_p},\mathrm{a},\mathrm{Ax_1},\frac{\mathrm{t}}{4\mathrm{q^n}}\right) * \mathcal{M}\left(\mathrm{Ax_0},\mathrm{Ax_p},\mathrm{Ax_1},\mathrm{b},\frac{\mathrm{t}}{4\mathrm{q^n}}\right) *$$
$$\mathcal{M}\left(\mathrm{Ax_0},\mathrm{Ax_1},\mathrm{a},\mathrm{b},\frac{\mathrm{t}}{4\mathrm{q^n}}\right) * \mathcal{M}\left(\mathrm{Ax_1},\mathrm{Ax_p},\mathrm{a},\mathrm{b},\frac{\mathrm{t}}{4\mathrm{q^n}}\right).$$

For any positive integer p and n in N, if $\mathrm{n} = 2\mathrm{k}, \mathrm{p} = 2\mathrm{m}$ or $\mathrm{n} = 2\mathrm{k}+1, \mathrm{p} = 2\mathrm{m}$, By observing that $\mathcal{M}\left(\mathrm{Ax_0},\mathrm{Ax_p},\mathrm{a},\mathrm{b},\frac{\mathrm{t}}{\mathrm{q^n}}\right) \to 1$ as $\mathrm{n} \to \infty$ for each positive integer p and n in N.

$\mathcal{M}(\mathrm{Az}, \mathrm{A}^2\mathrm{z}, \mathrm{a}, \mathrm{b}, \mathrm{qt}) \geq \mathcal{M}(\mathrm{Az}, \mathrm{AAz}, \mathrm{a}, \mathrm{b}, \mathrm{qt})$

$\geq \min\{\mathcal{M}(\mathcal{S}\mathrm{z}, \mathrm{AAz}, \mathrm{a}, \mathrm{b}, \mathrm{t}), \mathcal{M}(\mathcal{T}\mathrm{z}, \mathrm{Az}, \mathrm{a}, \mathrm{b}, \mathrm{t}), \mathcal{M}(\mathcal{T}\mathrm{Az}, \mathrm{Az}, \mathrm{a}, \mathrm{b}, \mathrm{t})\}$

$\geq \min\{\mathcal{M}(\mathcal{S}\mathrm{z}, \mathrm{A}\mathcal{T}\mathrm{z}, \mathrm{a}, \mathrm{b}, \mathrm{t}), \mathcal{M}(\mathrm{Az}, \mathrm{Az}, \mathrm{a}, \mathrm{b}, \mathrm{t}), \mathcal{M}(\mathrm{A}\mathcal{T}\mathrm{z}, \mathrm{Az}, \mathrm{a}, \mathrm{b}, \mathrm{t})\}$

$\geq \min\{\mathcal{M}(\mathcal{S}\mathrm{z}, \mathrm{A}\mathcal{T}\mathrm{z}, \mathrm{a}, \mathrm{t}), \mathcal{M}(\mathrm{Az}, \mathrm{Az}, \mathrm{a}, \mathrm{t}), \mathcal{M}(\mathrm{A}\mathcal{T}\mathrm{z}, \mathrm{Sz}, \mathrm{a}, \mathrm{t})\}$

$\geq \mathcal{M}(\mathcal{S}\mathrm{z}, \mathrm{A}\mathcal{T}\mathrm{z}, \mathrm{a}, \mathrm{b}, \mathrm{t}) \geq \mathcal{M}(\mathcal{S}\mathrm{z}, \mathrm{AAz}, \mathrm{a}, \mathrm{t}) \geq \mathcal{M}(\mathrm{Az}, \mathrm{A}^2\mathrm{z}, \mathrm{a}\ \mathrm{t}) \ldots \geq \mathcal{M}\left(\mathrm{Az}, \mathrm{A}^2\mathrm{z}, \mathrm{a}, \mathrm{b}, \frac{\mathrm{t}}{\mathrm{q}^\mathrm{n}}\right).$

The sequence $\{\mathrm{Ax}_\mathrm{n}\}$ is consequently Cauchy. There are $\mathrm{z} = \lim_{\mathrm{n}\to\infty} \mathrm{Ax}_\mathrm{n}$ and $\mathrm{z} = \lim_{\mathrm{n}\to\infty} \mathcal{S}\mathrm{x}_{2\mathrm{n}-1} = \lim_{\mathrm{n}\to\infty} \mathcal{T}\mathrm{x}_{2\mathrm{n}}$ since the space X is complete. As a result, $\mathrm{Az} = \mathcal{S}\mathrm{z} = \mathcal{T}\mathrm{z}$ and Given that $\lim_{\mathrm{n}\to\infty} \mathcal{M}\left(\mathrm{Az}, \mathrm{A}^2\mathrm{z}, \mathrm{a}, \mathrm{b}, \frac{\mathrm{t}}{2\mathrm{q}^\mathrm{n}}\right) = 1$ Consequently, $\mathrm{Az} = \mathrm{A}^2\mathrm{z}$. Z is therefore the common fixed point of A, $\mathcal{S}$, and $\mathcal{T}$.

Let w (w ≠ z) be another common fixed point of $\mathcal{S}$, $\mathcal{T}$, and A for uniqueness. Then, using (5), we write

$$\mathcal{M}(\mathrm{Az}, \mathrm{Aw}, \mathrm{a}, \mathrm{b}, \mathrm{qt}) \geq \frac{\min\{\mathcal{M}(\mathcal{S}\mathrm{z},\mathrm{Aw},\mathrm{a},\mathrm{b},\mathrm{t}),\mathcal{M}(\mathcal{T}\mathrm{z},\mathrm{Az},\mathrm{a},\mathrm{b},\mathrm{t}),\mathcal{M}(\mathcal{T}\mathrm{w},\mathrm{Az},\mathrm{a},\mathrm{b},\mathrm{t})\}}{\min\{\mathcal{M}(\mathcal{S}\mathrm{z},\mathrm{Aw},\mathrm{a},\mathrm{b},\mathrm{t}),\mathcal{M}(\mathcal{T}\mathrm{w},\mathrm{Az},\mathrm{a},\mathrm{b},\mathrm{t})\}}.$$

This indicates $\mathcal{M}(\mathrm{z}, \mathrm{w}, \mathrm{a}, \mathrm{b}, \mathrm{qt}) \geq \mathcal{M}(\mathrm{z}, \mathrm{w}, \mathrm{a}, \mathrm{b}, \mathrm{t})$.

Thus, we write $\mathrm{z} = \mathrm{w}$ using the lemma. The proof of *Theorem 3* is now complete.

4 | Conclusion

The main objective of this study was to introduce a new kind of contractive condition in order to expand the theory of fixed points in fuzzy 3-metric spaces. In contrast to previous fixed point results that depend on similarity-type or classical contraction criteria, our method adapts the condition to represent the characteristics of interactions between three elements in a fuzzy environment.

Using this contractive mapping, we demonstrate the existence and uniqueness of the fixed point under the suggested condition, thus proving a new fixed point theorem. The method of proof is based on building a convergent sequence with the fuzzy 3-metric and showing that it satisfies the fixed point requirements.

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Data Availability

The data supporting the findings of this study are available from the corresponding author upon reasonable request.

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