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Ramanujan Primes and Negative Pell's Equation

V. Sangeetha[1], **T. Anupreethi**[2,*], **Manju Somanath**[1]

[1]Department of Mathematics National College(Autonomous) (Affiliated to Bharathidasan University), Trichy - 620001,Tamil Nadu, India.prasansangee@gmail.com; manjuajil@yahoo.com.

[2]PG and Research Department of Mathematics, anupreethitamil@gmail.com.

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Abstract

The Diophantine equation under consideration to find the non-zero integral solution are of the type $x^2 = (Rp)y^2 - (R_1)^t$ which represents a very general type of negative Pell's equation formed using Ramanujan primes as coefficients. The equation are all formed using the first 10 Ramanujan primes 2,11,17,29,41,47,59,67,71 & 97. Fixing the 1^{st} Ramanujan prime $R_1 = 2$, we seek for solutions of Pell's equation with coefficients R_p, where R_p denotes the p^{th} Ramanujan prime.

MSC Classification Number: 11D09,11D99.

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1|Introduction

The Diophantine equation $x^2 - Dy^2 = 1$, also known as Pell's equation, is known to have an infinite number of integer solutions for any square - free positive integer $D > 1$ [2]. The solutions stem from a single fundamental solution found in Algebraic Number Theory. Brahmagupta and Bhaskara were the first to study the classical Pell's equation, and Lagrange developed the entire theory.There are an infinite number of integer solutions (x_n, y_n) to Pell's equation $x^2 - Dy^2 = \pm 1$ for $n \geq 1$. Since all other solutions to this equation can be (easily) derived from the first non-trivial positive integer solution (x_1, y_1) (in this case, x_1 or $x_1 + y_1\sqrt{D}$ is minimum), it is known as the fundamental solution. Let $x^2 - Dy^2 = 1$ have a fundamental solution (x_1, y_1). Then, for any integer $n \geq 2$,

Corresponding Author: anupreethitamil@gmail.com

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the n-th positive solution of this equation, (x_n, y_n), is defined by the equality $x_n + y_n\sqrt{D} = (x_1 + y_1\sqrt{D})^n$.The corresponding negative Pell's equation,

$$x^2 - Dy^2 = -1, \tag{1}$$

is considerably more mysterious. For (1) to be solvable in integers, it is required that $D \equiv 1$ or $2 \pmod 4$, and that all odd prime divisors of D have forms that are congruent to 1 modulo 4. These conditions, however, do not guarantee the existence of a solution. One may visit A031396 in the OEIS [16] for some known numbers D = 2, 5, 10, 13, 17, 26, 29, 37, 41, 50, ... such that (1) is solvable in integers x and y.For several decades, certain number theorists have been committed to developing standards for the solvability of (1). A few requirements are listed in [2, 10, 11, 17]. According to Newman's [11] demonstration, (1) can be solved in integers if $D = \prod_{i=1}^{r} p_i$, where $r = 2$ or r is odd, and p_i's are primes congruent to 1 modulo 4 and satisfy $\left(\frac{p_i}{p_j}\right) = -1$ for all $1 \le i, j \le r$ with $i \neq j$. The standard Jacobi symbol is $\left(\frac{\cdot}{\cdot}\right)$ in this case. Conversely, Mollin [10] found a relationship between $x^2 - Dy^2 = 1$ and the negative Pell's equation (1).
He demonstrated that (1) can be solved in integers x and y if and only if $x_0 \equiv -1 \pmod{2D}$ is satisfied by the fundamental solution (x_0, y_0) of $x^2 - Dy^2 = 1$. Despite the significance of these findings, their approaches are constrained if $D \gg 1$. For instance, there are numerous situations where the fundamental solutions are fairly large. Thus, extensive computations are required to verify these conditions. Positive Pell's equations have recently been found to have well-known solvability criteria. A primitive Pythagorean triple for D can be found by using the length of its period as a guide. Another method is to compute the simple continued fraction of $\sqrt{D}$.[18-20]
A prime number that satisfies a result about the prime-counting function proved by Srinivasa Ramanujan is known as a Ramanujan prime in mathematics. As he points out, Bertrand's postulate was initially proven by Chebyshev. Ramanujan presented a revised demonstration of the concept in 1919. Ramanujan arrived to a generalized result at the end of the two-page published work. Consequently, 2, 11, 17, 29, and 41 are the initial five Ramanujan primes.

Take note that R_n must be a prime number in order for it to exist:$\pi(x) - \pi\left(\frac{x}{2}\right)$ and, hence,$\pi(x)$ must increase by obtaining another prime at $x = R_n$. Since $\pi(x) - \pi\left(\frac{x}{2}\right)$can increase by atmost 1,$\pi(R_n) - \pi\left(\frac{R_n}{2}\right) = n$. In this paper the negative Pell's equation $x^2 = (R_p)y^2 - (R_1)^t$ formed using the p^{th} Ramanujan prime R_p, are analysed for obtaining non-zero integer solution for various choice of t and are illustrated for primes less then 100.

2|Preliminaries

Theorem 0.1. *If x_1, y_1 is the fundamental solution of $x^2 - Dy^2 = 1$,then every positive solution of the equation is given by x_n, y_n where x_n and y_n are the integers determined from $x_n + y_n\sqrt{D} = (x_1 + y_1\sqrt{D})^n, n = 1, 2, 3 \cdots$*

For illustration,consider the fundamental solution $x_1 = 24$ & $y_1 = 5$ of
$x^2 - 23y^2 = 1$. A second positive solution (x_2, y_2) can be obtained from the formula

$$x_2 + y_2\sqrt{23} = (24 + 5\sqrt{23})^2 = 1151 + 240\sqrt{23}$$

This gives $x_2 = 1151, y_2 = 240$.These integers also satisfy the equation
$x^2 - 23y^2 = 1$,since $1151^2 - 23y^2 = 1$.
$\therefore$ Any positive solution can be calculated using the Theorem 2.1.

Theorem 0.2. *Suppose that the equation*

$$x^2 - Dy^2 = -1 \tag{2}$$

has solution in positive integers and (x_0, y_0) be its minimal solution.The general solution to (2) is given by $(x_n, y_n)_{n\ge 0}$ where $x_n = x_0u_n + Dy_0v_n$,$y_n = y_0u_n + x_0v_n$ and $(u_n, v_n)_{n\ge 0}$ is the general solution to Pell's equation $u^2 - Dv^2 = 1$.

Theorem 0.3. *Let p be a prime. The negative Pell's equation $x^2 - py^2 = -1$ is solvable if and only if $p = 2$ or $p \equiv 1 \pmod 4$.*

2.1 Testing the solubility of the negative Pell's equation

Suppose D is a positive integer and not a perfect square. Then the negative Pell's equation $x^2 - Dy^2 = -1$ is soluble if and only if D is expressible as
$D = a^2 + b^2, gcd(a, b) = 1$, a and b positive, b odd and the Diophantine equation $-bv^2 + 2avw + bw^2 = 1$ has a solution(the case of solubility occurs for exactly one such (a,b)) .

The Algorithm

(2.1.1). Find all expressions of D as a sum of two relatively prime squares using cornacchia's method.If none exist, the negative Pell's equation is not soluble.

(2.1.2). For each representation $D = a^2 + b^2$,$gcd(a, b) = 1$, a and b positive, b odd,test the solubility of $-bv^2 + 2avw + bw^2 = 1$ using Lagrange-Matthews algorithm.If solutions exist,the negative Pell's equation is soluble.

(2.1.3). If each representation yields no solution,then the negative Pell's equation is insoluble.

3|Main Results

The equation under consideration in the negative Pell's equation $x^2 = (Rp)y^2 - (R_1)^t$, where R_p denotes the p^{th} Ramanujan prime and $R_1 = 2$,the first Ramanujan Prime.

System-I: Diophantine equation $x^2 = 11y^2 - 2^t$,with $R_2 = 11$ and $R_1 = 2$.

This system concerns with negative Pell's equation $x^2 = 11y^2 - 2^t, t \in N$.Here the prime $R_2 = 11$ cannot be expressed as the sum of two relatively prime squares.Thus by the condition (2.1.1) of the algorithm,the negative Pell's equation $x^2 - 11y^2 = -1$ is not soluble. Thus we can conclude that the negative Pell's equation $x^2 = 11y^2 - 2^t$ is insoluble.

System-II: Diophantine Equation $x^2 = 17y^2 - 2^t$ with $R_3 = 17, R_1 = 2$.

The prime $R_3 = 17$ satisfies all the conditions of Theorem 2.3,and hence we conclude that the negative Pell's equation $x^2 = 17y^2 - 2^t$,$t \in N$ is solvable and infinitely many positive integer solutions are obtained for the various choices of t.The more general Pell's equation

$$u^2 = 17v^2 + 1 \tag{I}$$

can be viewed and its general solution (u_n, v_n) is given by

$$u_n = \frac{1}{2}f_n;\ v_n = \frac{1}{2\sqrt{17}}g_n, \tag{II}$$

where $f_n = (33 + 8\sqrt{17})^n + (33 - 8\sqrt{17})^n$
$g_n = (33 + 8\sqrt{17})^n - (33 - 8\sqrt{17})^n$

Choice-I: t is an even number

<u>Case 1:</u>t=2,

The Pell's equation under consideration is

$$x^2 = 17y^2 - 4 \tag{3}$$

with the initial solution $x_0 = 8; y_0 = 2$.Applying Theorem 2.2, a series of non-zero integer solutions to (3) are discovered as
$x_n = \frac{1}{2}[8f_n + 2\sqrt{17}g_n]$
$y_n = \frac{1}{2\sqrt{17}}[2\sqrt{17}f_n + 8g_n]$,where f_n and g_n are defined as in (II)

The recurrence relations satisfied by the solutions of (3) are given by
$x_{n+2} - 66x_{n+1} + x_n = 0$
$y_{n+2} - 66y_{n+1} + y_n = 0$

<u>Case 2:</u>$t = 2k + 2$, where $1 \le k \le 7$,

The Pell's equation thus obtained is

$$x^2 = 17y^2 - 2^{2k+2} \tag{4}$$

with initial solution $x_0 = 2^{k-1}, y_0 = 2^{k-1}$.Applying Brahmagupta lemma between the solutions(x_0, y_0) and (u_n, v_n) from (II),the sequence of non-zero distinct integer solutions to (4) are obtained as

$x_n = \frac{2^{k-1}}{2}[f_n + \sqrt{17}g_n]$

$y_n = \frac{2^{k-1}}{2\sqrt{17}}[\sqrt{17}f_n + g_n]$.

Case 3: t=2k+16, $k \in N$

Consider

$$x^2 = 17y^2 - 2^{2k+16} \tag{5}$$

with initial solution $x_0 = 59(2^{k-1}), y_0 = 125(2^{k-1})$

Applying Theorem 2.2, and using (II) we get infinite number of solutions to (5) as

$x_n = \frac{2^{k-1}}{2}[59f_n + 125\sqrt{17}g_n]$

$y_n = \frac{2^{k-1}}{2\sqrt{17}}[125\sqrt{17}f_n + 59g_n]$.

Choice-II: t is an odd number

Case 1: $t = 2k + 1, 1 \leq k \leq 4$

The Pell's equation to be solved is

$$x^2 = 17y^2 - 2^{2k+1} \tag{6}$$

The following table represents various values of t with the initial & general solutions to (6).

$t = 2k + 1$	(x_0, y_0)	(x_n, y_n)
$k = 1$	(3,1)	$x_n = \frac{1}{2}[3f_n + \sqrt{17}g_n]$; $y_n = \frac{1}{2\sqrt{17}}[\sqrt{17}f_n + 3g_n]$
$k = 2$	(6,2)	$x_n = \frac{1}{2}[6f_n + 2\sqrt{17}g_n]$; $y_n = \frac{1}{2\sqrt{17}}[2\sqrt{17}f_n + 6g_n]$
$k = 3$	(5,3)	$x_n = \frac{1}{2}[5f_n + 3\sqrt{17}g_n]$; $y_n = \frac{1}{2\sqrt{17}}[3\sqrt{17}f_n + 5g_n]$
$k = 4$	(10,6)	$x_n = \frac{1}{2}[10f_n + 6\sqrt{17}g_n]$; $y_n = \frac{1}{2\sqrt{17}}[6\sqrt{17}f_n + 10g_n]$

Case 2: t=2k+9, $k \in N$
The Pell's equation thus obtained is

$$x^2 = 17y^2 - 2^{2k+9} \tag{7}$$

Let (x_0, y_0) be the initial solution of (7)given by $x_0 = 3(2^{k-1}), y_0 = 11(2^{k-1})$.
Applying Theorem 2.2, a series of non-zero integer solution to (7) are discovered as
$x_n = \frac{2^{k-1}}{2}[3f_n + 11\sqrt{17}g_n]$
$y_n = \frac{2^{k-1}}{2\sqrt{17}}[11\sqrt{17}f_n + 3g_n]$, where f_n and g_n are defined as in (II).
The recurrence relations satisfied by the solutions of (7) are given by
$x_{n+2} - 66x_{n+1} + x_n = 0$
$y_{n+2} - 66y_{n+1} + y_n = 0$
System -III: Diophantine Equation $x^2 = 29y^2 - 2^t$ with $R_4 = 29, R_1 = 2$.
Given that $R_4 = 29$ satisfies all the settings of Theorem 2.3,we can conclude that the negative Pell's equation $x^2 = 29y^2 - 2^t$ is solvable in integers.
Consider the case when t is an even number.
In this case,the Pell's equation is

$$x^2 = 29y^2 - 2^{2k} \tag{8}$$

Let (x_0, y_0) be the primary solution of (8) specified by $x_0 = 5(2^{k-1}), y_0 = 2^{k-1}$. Analysing the pell's equation $u^2 = 29v^2 + 1$ to discover the additional solutions to (8), whose generic solution (u_n, v_n) is provided by
$u_n = \frac{1}{2}f_n;\ v_n = \frac{1}{2\sqrt{29}}g_n$
where $f_n = (9801 + 1820\sqrt{29})^n + (9801 - 1820\sqrt{29})^n$
$g_n = (9801 + 1820\sqrt{29})^n - (9801 - 1820\sqrt{29})^n$
Applying Brahmagupta lemma between the solutions(x_0, y_0) and (u_n, v_n),the sequence of non-zero distinct integer solutions to (8) are obtained as
$x_n = \frac{2^{k-1}}{2}[5f_n + \sqrt{29}g_n]$
$y_n = \frac{2^{k-1}}{2\sqrt{29}}[\sqrt{29}f_n + 5g_n]$.
The recurrence relations fulfilled by the solutions of (8) are specified by
$x_{n+2} - 1960x_{n+1} + x_n = 0$
$y_{n+2} - 1960y_{n+1} + y_n = 0$
Observation: This system fails to generate integer solutions for all the odd choices of t.
System-IV: Diophantine Equation $x^2 = 41y^2 - 2^t$ with $R_5 = 41, R_1 = 2$.
For this particular equation,we consider the prime 41.Using the Algorithm and testing with
$(a, b) = (4, 5)$,$-bv^2 + 2avw + bw^2 = 1$ has a solution $(v, w) = (2, 1)$.Therefore,we can develop this discussion by confirming that the negative pell equation $x^2 = 41y^2 - 2^t.t \in N$ is solvable in integers.
Consider the pellian equation

$$u^2 = 41v^2 + 1 \tag{III}$$

whose general solution (u_n, v_n) is given by

$$u_n = \frac{1}{2}f_n;\ v_n = \frac{1}{2\sqrt{41}}g_n \tag{IV}$$

where $f_n = (2049 + 320\sqrt{41})^n + (2049 - 320\sqrt{41})^n$
$g_n = (2049 + 320\sqrt{41})^n - (2049 - 320\sqrt{41})^n$
Choice-1: t is an even number
case 1: t=2
The Pell's equation to be checked for its solutions is

$$x^2 = 41y^2 - 4 \tag{9}$$

Assume (x_0, y_0) be the fundamental solution of (9) given by $x_0 = 64, y_0 = 10$
Applying Theorem 2.2, a collection of nonzero different integer solutions to (9) is achieved as

$x_n = \frac{1}{2}[64f_n + 10\sqrt{41}g_n]$
$y_n = \frac{1}{2\sqrt{41}}[10\sqrt{41}f_n + 64g_n].$
The recurrence relation derived by employing the solutions are
$x_{n+2} - 4098x_{n+1} + x_n = 0$
$y_{n+2} - 4098y_{n+1} + y_n = 0.$
Case2: $t = 2k + 2, 1 \leq k \leq 3$
Consider

$$x^2 = 41y^2 - 2^{2k+2} \tag{10}$$

Let (x_0, y_0) represent the first solution to (10) proposed by $x_0 = 5(2^{k-1}), y_0 = 2^{k-1}$
Applying Brahmagupta Lemma between (x_0, y_0) and (u_n, v_n) from (IV),the series of different non-zero integer solutions are discovered as
$x_n = \frac{2^{k-1}}{2}[5f_n + \sqrt{41}g_n]$
$y_n = \frac{2^{k-1}}{2\sqrt{41}}[\sqrt{41}f_n + 5g_n].$
Case 3: $t = 2k + 8, k \in N$
The Pell's equation is

$$x^2 = 41y^2 - 2^{2k+8} \tag{11}$$

Let (x_0, y_0) represent the initial solutions to (11) offered by $x_0 = 2^{k-1}, y_0 = 5(2^{k-1})$
Applying Brahmagupta Lemma between (x_0, y_0) and (u_n, v_n) from (IV),the sequence of non-zero distinct integer solutions to (11) are obtained as
$x_n = \frac{2^{k-1}}{2}[f_n + 5\sqrt{41}g_n]$
$y_n = \frac{2^{k-1}}{2\sqrt{41}}[5\sqrt{41}f_n + g_n].$
Choice-2: t is an odd number
Case 1: t=3
Consider

$$x^2 = 41y^2 - 2^3 \tag{12}$$

Applying Theorem 2.2 ,a collection of nonzero different integer solutions to (12) are achieved as
$x_n = \frac{1}{2}[19f_n + 3\sqrt{41}g_n]$
$y_n = \frac{1}{2\sqrt{41}}[3\sqrt{41}f_n + 19g_n].$
Case 2: $t = 2k + 3, k \in N$
The Pell's equation to be examined is

$$x^2 = 41y^2 - 2^{2k+3} \tag{13}$$

Applying Brahmagupta Lemma between (x_0, y_0) and (u_n, v_n) from (IV),the sequence of non-zero distinct integer solutions to (13) are obtained as
$x_n = \frac{2^{k-1}}{2}[3f_n + \sqrt{41}g_n]$
$y_n = \frac{2^{k-1}}{2\sqrt{41}}[\sqrt{41}f_n + 3g_n].$
System-V
In this system, we consider the Ramanujan primes $R_6 = 47, R_7 = 59, R_8 = 67$ and $R_9 = 71$.We form Pell's equations $x^2 = 47y^2 - 2^t, x^2 = 59y^2 - 2^t, x^2 = 67y^2 - 2^t$ and $x^2 = 71y^2 - 2^t$.All these primes fail to satisfy the conditions of Theorem 2.3 and thus we conclude that, the above equations so formed does not possess integer solutions.
System-VI Diophantine Equation $x^2 = 97y^2 - 2^t$ with $R_{10} = 97, R_1 = 2$.
We consider the prime 97 in support of this precise equation. Given that 97 meets every requirement of Theorem

2.3,we can come to the conclusion that negative the Pell's equation $x^2 = 97y^2 - 2^t$ is solvable.The more general Pell's equation

$$u^2 = 97v^2 + 1 \tag{V}$$

can be viewed and its general solution (u_n, v_n) is given by

$$u_n = \frac{1}{2} f_n; \; v_n = \frac{1}{2\sqrt{97}} g_n, \tag{VI}$$

where $f_n = (62809633 + 6377352\sqrt{97})^n + (62809633 - 6377352\sqrt{97})^n$
$g_n = (62809633 + 6377352\sqrt{97})^n - (62809633 - 6377352\sqrt{97})^n$

Choice-1:t is an even number.
Case 1: t=2
The Pell's equation so generated is

$$x^2 = 97y^2 - 4 \tag{14}$$

Let (x_0, y_0) represent the initial solution to (14) offered by $x_0 = 11208, y_0 = 1138$.Applying Brahmagupta Lemma between (x_0, y_0) and (u_n, v_n) from (VI),the sequence of non-zero distinct integer solutions to (14) are obtained as
$x_n = \frac{1}{2}[11208 f_n + 1138\sqrt{97} g_n]$
$y_n = \frac{1}{2\sqrt{97}}[1138\sqrt{97} f_n + 11208 g_n]$.
Case 2: $t = 2k + 2, k = 1, 2, 3, 4, 5$
Consider

$$x^2 = 97y^2 - 2^{2k+2} \tag{15}$$

The sequence of non-zero distinct integer solutions to (15) are obtained as
$x_n = \frac{2^{k-1}}{2}[9 f_n + \sqrt{97} g_n]$
$y_n = \frac{2^{k-1}}{2\sqrt{97}}[\sqrt{97} f_n + 9 g_n]$.
Case 3: t=14
Using Theorem 2.2,the general solution to $x^2 = 97y^2 - 2^{14}$ is given by
$x_n = \frac{1}{2}[3 f_n + 13\sqrt{97} g_n]$
$y_n = \frac{1}{2\sqrt{97}}[13\sqrt{97} f_n + 3 g_n]$.
Case 4: $t = 2k + 14, k \in N$
The Pell's equation is

$$x^2 = 97y^2 - 2^{2k+14} \tag{16}$$

has solutions in positive integer and let $(6(2^{k-1}), 26(2^{k-1}))$ be its minimal solution.The general solution to (16)is given by (x_n, y_n) where
$x_n = \frac{2^{k-1}}{2}[6 f_n + 26\sqrt{97} g_n]$
$y_n = \frac{2^{k-1}}{26\sqrt{97}}[26\sqrt{97} f_n + 6 g_n]$.
Choice-2:t is an odd number
Case 1: t=3,5,7
The Pell's equation is

$$x^2 = 97y^2 - 2^t \tag{17}$$

The following table represents various values of t with the initial & general solutions to (17).

Case 2: $t = 2k + 7, k \in N$
The Pell's equation is

$$x^2 = 97y^2 - 2^{2k+7} \tag{18}$$

t	(x_0, y_0)	(x_n, y_n)
3	(325,33)	$x_n = \frac{1}{2}[325 f_n + 33\sqrt{97} g_n]$; $y_n = \frac{1}{2\sqrt{97}}[33\sqrt{97} f_n + 325 g_n]$
5	(29,3)	$x_n = \frac{1}{2}[29 f_n + 3\sqrt{97} g_n]$; $y_n = \frac{1}{2\sqrt{97}}[3\sqrt{97} f_n + 29 g_n]$
7	(58,6)	$x_n = \frac{1}{2}[58 f_n + 6\sqrt{97} g_n]$; $y_n = \frac{1}{2\sqrt{97}}[6\sqrt{97} f_n + 58 g_n]$

with initial solution $x_0 = 19(2^{k-1})$,$y_0 = 3(2^{k-1})$.Applying Brahmagupta lemma between (x_0, y_0) and (u_n, v_n) from (VI), the progression of non-zero different integer solutions to (18) is found to be

$$x_n = \frac{2^{k-1}}{2}[19 f_n + 3\sqrt{97} g_n]$$

$$y_n = \frac{2^{k-1}}{2\sqrt{97}}[3\sqrt{97} f_n + 19 g_n].$$

4|Conclusion

In this paper, the equations are all formed restricting the primes less than 100. It can be extended to any bound and also to any prime number and thus delve into the world of Pell's equations.

5|Author Contribution

Credit authorship contribution statement:

Manju Somanath: Visualization, Validation,Conceptualisation.
V.Sangeetha: Writing – original draft, Validation, Data curation.
T.Anupreethi: Writing – original draft, Visualization,Validation.

Declaration of competing interest:

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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