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Solutions of Pell's Equation Involving Left Truncated Primes

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Abstract

A Left-truncatable prime is a prime which in a given base (say 10) does not contain 0 and which remains prime when the leading (left) digit is successively removed. For example, 317 is left-truncatable prime since 317, 17 and 7 are all prime. Taking the cue from this initial research, we attempt to find the possible solutions for the Pell's equation $x^2 = 137y^2 - 37^m$ for all choices of $m \in \mathbb{N}$. In this paper, we focused primarily on Pell's equations involving the left-truncatable primes and present to you another mysterious series and pattern typically associated with the Pell's equation. As we proceed through the research, we will bring to the fore the recurrence relations among the identified solutions.

Keywords: Pell's equation, Diophantine equations, Integer solutions, Recurrence relation, Left truncated primes.

1 | Introduction

"Number Theory" has time and again thrown its magic at the human world, specifically the world of wizards; the Mathematicians. We all know about the beautiful instances of the prime numbers. In this article, we explore the realm of truncated primes. We are talking about left truncated primes. The initial thought was given to these patterns by "Leslie Card" who first started off by researching into right truncated primes. This was picked up later and subsequent research was done to understand the left truncated primes too. It threw light on the specific part of prime numbers and was given to understand that we have 4260 left truncated primes. The following sequence of left truncatable primes will enable us to understand the magic of these numbers. These numbers are 2, 3, 5, 7, 13, 17, 23, 37, 43, 47, 53, 67, 73, 83, 97, 113, 137, 167, 173, 197, 223, 283, 313, 317, 337, 347, 353, 367, 373, 383, 397, 443, 467, 523, 547, 613, 617, 643, 647, 653, 673, 683, 743,

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773, 797, 823, 853, 883, 937, 947, 953, 967, 983, 997, ... As noted in the sequence, we have another manifestation of the Pell's equation which is the prime focus of this research. We took up the left truncatable primes and attempted to find solutions for the equation represented by this prime. In this instance, we identified the Pell's equation $x^2 = 137y^2 - 37^m, m \in \mathbb{N}$; using the left truncated prime numbers 137 and 37. Note that 137 is the initial prime number, where upon truncating the left most digit; we get 37. The equation is built on these two primes 137 and 37. Our attempt will be to search for its non-trivial integer solutions. To derive the solutions, we will set the proof rolling by we approached the quest with the case of choices of m generalized in all even and odd integers [1–3].

2 | Preliminaries

Theorem 1. If x_1 , y_1 is considered as the fundamental solution of $x^2 - dy^2 = 1$. Then to be noted is that every positive solution of the equation is given by x_n, y_n where x_n and y_n are the integers determined from $x_n + y_n\sqrt{d} = \left(x_1 + y_1\sqrt{d}\right)^n$, for $n = 1,2,3, ...$ [4].

2.1 | Solubility of the Negative Pell's Equation-Our Test Approach

We assume that D is a positive integer, and considered not a perfect square. Then the negative Pell's equation $x^2 - Dy^2 = -1$ is considered soluble if and only if D is expressed as $D = a^2 + b^2$, $\gcd(a, b) = 1$, a and b are positive, b is odd and the Diophantine equation $-bV^2 + 2aVW + bW^2 = 1$ has a solution [5]. (We highlight this as the case of solubility that occurs for exactly one such (a, b)) [6].

The Algorithm followed by us is illustrated below:

I. We will first find all expressions of D considered as a sum of two relatively prime squares using Cornacchia's method. If none exists - the negative Pell's equation is not soluble.

II. For each representation D = $a^2 + b^2$, gcd (a, b) = 1, a and b positive, b odd, we will test the solubility of $-bV^2 + 2aVW + bW^2 = 1$ using the Lagrange-Matthews algorithm. If soluble and it exists - the negative Pell's equation is soluble.

III. If each representation yields no probable solution, then the negative Pell's equation is insoluble [7].

Theorem 2. If p is a prime number, the negative Pell's equation $x^2 - py^2 = -1$ is considered solvable if and only if $p = 2$ or $p \equiv 1 (\mathrm{mod} 4)$.

3 | Method of Analysis

Theorem 3. The negative Pell's equation $x^2 - 137y^2 = -37^m, m \in \mathbb{N}$ is solvable in integers.

Proof. Consider the negative Pell's equation $x^2 = 137y^2 - 37^m, m \in \mathbb{N}$. For the negative Pell's equation, we will consider the prime $p = 137$, which satisfies the identified conditions of *Theorem 2*. Therefore, equation $x^2 - 137y^2 = -1$, is solvable and we can substantiate with certainty the proof that the negative Pell's equation $x^2 = 137y^2 - 37^m, m \in \mathbb{N}$ is solvable and prevalent in integers [4], [8].

Using the Algorithm as illustrated in solubility of the negative Pell's equation for $(a, b) = (4,11)$: $-bV^2 + 2aVW + bW^2 = 1$ has a solution $(V, W) = (-7, 10)$, hence the negative Pell's equation $x^2 - 137y^2 = -1$ is soluble.

Choice 1. $m = 1$

The Pell's equation in focus is

$$x^2 = 137y^2 - 37. \quad \textbf{(1)}$$

Let (x_0, y_0) be the initial solution of *Eq. (1)* given by $x_0 = 10,\ y_0 = -1$.

In our quest to find the other solutions of *Eq. (1)*, consider the generalized form of the Pell's equation

$$x^2 = 137y^2 + 1. \quad (2)$$

The initial solution of *Eq. (2)* is (6083073,519712) and the general solution $(\widetilde{x_n}, \widetilde{y_n})$ given by *Theorem 1* as $\widetilde{x_n} = \frac{1}{2} f_n$, $\widetilde{y_n} = \frac{1}{2\sqrt{137}} g_n$, where $f_n = (6083073 + 519712\sqrt{137})^{n+1} + (6083073 - 519712\sqrt{137})^{n+1}$, $g_n = (6083073 + 519712\sqrt{137})^{n+1} - (6083073 - 519712\sqrt{137})^{n+1}, n = 0,1,2 \dots$.

By applying Brahma Gupta lemma between (x_0, y_0) and $(\widetilde{x_n}, \widetilde{y_n})$ the possible sequence of non-zero distinct integer solutions to *Eq. (1)* are obtained as given below

$$x_{n+1} = x_0\widetilde{x_n} + dy_0\widetilde{y_n}, \quad y_{n+1} = x_0\widetilde{y_n} + y_0\widetilde{x_n}, \quad (3)$$

$$x_{n+1} = \frac{1}{2}\left[10f_n - \sqrt{137}g_n\right], \; y_{n+1} = \frac{1}{2\sqrt{137}}\left[-\sqrt{137}f_n + 10g_n\right]. \quad (4)$$

Also to be noted is the recurrence relation satisfied by the solution of *Eq. (1)* given by

$$x_{n+2} - 12166146x_{n+1} + x_n = 0,$$
$$y_{n+2} - 12166146y_{n+1} + y_n = 0. \quad (5)$$

Choice 2. m = 3.

The Pell's equation is

$$x^2 = 137y^2 - 50653. \quad (6)$$

Let (x_0, y_0) be the initial solution of *Eq. (6)* given by $x_0 = 370, \; y_0 = -37$. Applying Brahma Gupta lemma between (x_0, y_0) and $(\widetilde{x_n}, \widetilde{y_n})$ the possible sequence of non-zero distinct integer solutions to *Eq. (6)* are obtained by equation *Eq. (3)* as given below

$$x_{n+1} = \frac{1}{2}\left[370f_n - 37\sqrt{137}g_n\right],$$
$$y_{n+1} = \frac{1}{2\sqrt{137}}\left[-37\sqrt{137}f_n + 3700g_n\right]. \quad (7)$$

The recurrence relation satisfied by the solution of *Eq. (6)* are given by the equations below

$$x_{n+2} - 12166146x_{n+1} + x_n = 0,$$
$$y_{n+2} - 12166146y_{n+1} + y_n = 0. \quad (8)$$

Choice 3. m = 5.

The Pell's equation in focus is

$$x^2 = 137y^2 - 69343957. \quad (9)$$

Let (x_0, y_0) be the initial solution of *Eq. (9)* given by $x_0 = 13690, \; y_0 = -1369$.

Applying Brahma Gupta lemma between (x_0, y_0) and $(\widetilde{x_n}, \widetilde{y_n})$ the possible sequence of non-zero distinct integer solutions to *Eq. (9)* obtained by equation *Eq. (3)* as

$$x_{n+1} = \frac{1}{2}\left[13690f_n - 1369\sqrt{137}g_n\right],$$
$$y_{n+1} = \frac{1}{2\sqrt{137}}\left[-1369\sqrt{137}f_n + 13690g_n\right]. \quad (10)$$

The recurrence relation satisfied by the solution of *Eq. (9)* are given by the equations below

$$x_{n+2} - 12166146x_{n+1} + x_n = 0,$$
$$y_{n+2} - 12166146y_{n+1} + y_n = 0. \quad (11)$$

Choice 4. $m = 2k, k \in \mathbb{N}$.

The Pell's equation is

$$x^2 = 137y^2 - 37^{2k}, k \in \mathbb{N}. \quad (12)$$

Let (x_0, y_0) be the initial solution of equation *Eq. (12)* given by $x_0 = 1744(37)^k; y_0 = 149(37)^k$.

Applying Brahma Gupta lemma between (x_0, y_0) and $(\widetilde{x_n}, \widetilde{y_n})$ the possible sequence of non-zero distinct integer solutions to *Eq. (12)* are obtained by *Eq. (3)* as given below

$$x_{n+1} = \frac{37^k}{2}\left[1744f_n - 149\sqrt{137}g_n\right],$$
$$y_{n+1} = \frac{37^k}{2\sqrt{137}}\left[-149\sqrt{137}f_n + 1744g_n\right]. \quad (13)$$

The recurrence relation satisfied by the solution of *Eq. (12)* are given by the equations below

$$x_{n+2} - 12166146x_{n+1} + x_n = 0,$$
$$y_{n+2} - 12166146y_{n+1} + y_n = 0. \quad (14)$$

Choice 5. $m = 2k + 5, k \in \mathbb{N}$.

The Pell's equation is

$$x^2 = 137y^2 - 37^{2k+5}, k \in \mathbb{N}. \quad (15)$$

Let (x_0, y_0) be the initial solution of the *Eq. (15)* given by $x_0 = 506530(37)^{k-1}, y_0 = 50653(37)^{k-1}$.

Applying Brahma Gupta lemma between (x_0, y_0) and $(\widetilde{x_n}, \widetilde{y_n})$ the sequence of non-zero distinct integer solutions to *Eq. (15)* are obtained by *Eq. (3)* as

$$x_{n+1} = \frac{37^{k-1}}{2}\left[506530f_n - 50653\sqrt{137}g_n\right],$$
$$y_{n+1} = \frac{37^{k-1}}{2\sqrt{137}}\left[-50653\sqrt{137}f_n + 506530g_n\right]. \quad (16)$$

The recurrence relation satisfied by the solution of *Eq. (15)* are given by the equations below

$$x_{n+2} - 12166146x_{n+1} + x_n = 0,$$
$$y_{n+2} - 12166146y_{n+1} + y_n = 0. \quad (17)$$

4 | Conclusion

We conclude this research of solving a Pell's equation involving the left truncatable primes 137 and 37. As seen above illustrated by the analysis and the steps thereof; we have successfully proved the equation $x^2 = 137y^2 - 37^m, m \in \mathbb{N}$ for all integers. Finding the solution for the Pell's equation constituted by having left truncated primes reinforced the possibility of having solutions to our identified equation [9], [10].

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Data Availability

This study is based on theoretical analysis and does not involve any experimental or empirical data. All information supporting the findings is included within the article. Further clarifications can be obtained from the corresponding author upon reasonable request.

Conflicts of Interest

The authors affirm that there are no known conflicts of interest associated with this publication.

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