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Hyperfuzzy and SuperHyperfuzzy Extensions of Linear Programming: Models and Mathematical Foundations

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Abstract

A fuzzy set assigns to each element of a universe a membership degree within the interval [01] , thereby modeling imprecision and vagueness. A hyperfuzzy set extends this concept by associating each element with a nonempty subset of [01] , capturing both uncertainty and variability through a range of possible membership degrees. Building on this, a superhyperfuzzy set generalizes the framework further by assigning to each nonempty element in the= th power-set hierarchy a nonempty subset of [01] , thus enabling the representation of recursively structured and hierarchical uncertainty.

Linear programming is an optimization technique that aims to maximize or minimize a linear objective function subject to a set of linear equality and inequality constraints. Fuzzy linear programming generalizes this framework by incorporating fuzzy numbers into the objective coefficients and constraints, allowing for uncertainty in both parameters and feasible regions.

In this paper, we propose mathematical models for Hyperfuzzy Linear Programming and Superhyperfuzzy Linear Programming, and briefly examine their theoretical properties. We hope that these models will provide a foundation for further validation, development, and refinement in future research.

Keywords: Fuzzy set, Hyperfuzzy set, Superhyperfuzzy set, Fuzzy linear programming, Linear programming.

1|Preliminaries

This section provides an overview of the fundamental concepts and definitions essential for the discussions in this paper. In this paper, all sets are assumed to be finite.

1.1|Hyperfuzzy Set and the SuperHyperfuzzy Set

A fuzzy set assigns to each element in a universe a membership degree in the interval [01] , providing a mathematical framework for representing imprecision and vagueness [1–4].

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Several advanced frameworks have been developed to extend or complement the classical fuzzy set theory. These include:

- **Intuitionistic Fuzzy Sets** [5–7], which assign both a membership and a non-membership degree to each element;
- **Vague Sets** [8–10], which refine the interpretation of uncertainty through intervals;
- **Bipolar Fuzzy Sets** [11–13], allowing positive and negative degrees of membership;
- **Neutrosophic Sets** [14–16], which incorporate degrees of truth, indeterminacy, and falsity;
- **Neutrosophic Offsets** [17, 18], which model deviation or offset from standard membership structures;
- **Picture Fuzzy Sets** [19–21], which include positive, neutral, and negative membership components;
- **Hesitant Fuzzy Sets** [22–24], which allow a set of possible membership values for each element;
- **Quadri-Partitioned Neutrosophic Sets** [25–27], which divide uncertainty into four interpretive components;
- **Hyperneutrosophic Sets** [28], which introduce a higher-order hierarchical structure to neutrosophic models;
- **Plithogenic Sets** [29–33], which generalize fuzzy sets by incorporating attribute contradiction degrees.

In this paper, we focus on the *Hyperfuzzy Set* [34–37] and its recursive generalization, the *SuperHyperfuzzy Set* [28, 38], both of which aim to capture more complex forms of uncertainty by introducing set-valued and hierarchical membership degrees.

Definition 0.1 (Base Set)**.** A *base set* S is the foundational set from which complex structures such as powersets and hyperstructures are derived. It is formally defined as:

$$S = \{x \mid x \text{ is an element within a specified domain}\}.$$

All elements in constructs like $\mathcal{P}(S)$ or $\mathcal{P}_n(S)$ originate from the elements of S.

Definition 0.2 (Powerset)**.** The *powerset* of a set S, denoted $\mathcal{P}(S)$, is the collection of all possible subsets of S, including both the empty set and S itself. Formally, it is expressed as:

$$\mathcal{P}(S) = \{A \mid A \subseteq S\}.$$

Definition 0.3 (n-th Powerset)**.** (cf. [39–42])

The n-th powerset of a set H, denoted $\mathcal{P}_n(H)$, is defined iteratively, starting with the standard powerset. The recursive construction is given by:

$$\mathcal{P}_1(H) = \mathcal{P}(H), \quad \mathcal{P}_{n+1}(H) = \mathcal{P}(\mathcal{P}_n(H)), \quad \text{for } n \geq 1.$$

Similarly, the n-th non-empty powerset, denoted $\mathcal{P}_n^*(H)$, is defined recursively as:

$$\mathcal{P}_1^*(H) = \mathcal{P}^*(H), \quad P_{n+1}^*(H) = \mathcal{P}^*(\mathcal{P}_n^*(H)).$$

Here, $\mathcal{P}^*(H)$ represents the powerset of H with the empty set removed.

Example 0.4 (Real-World Example of n-th Powerset)**.** Let H be the set of basic ingredients for a simple recipe:

$$H = \{\text{tomato, cheese, basil}\}.$$

Then the first powerset

$$\mathcal{P}_1(H) = P(H)$$

is the collection of all possible ingredient combinations (including the empty set), for instance $\{\text{tomato, basil}\}$ or $\{\text{cheese}\}$, each of which you can think of as a "recipe."

The second powerset

$$\mathcal{P}_2(H) = \mathcal{P}\big(\mathcal{P}_1(H)\big)$$

is the set of all possible "menus," i.e. collections of recipes. For example,

$$\big\{\{\text{tomato, basil}\}, \{\text{cheese}\}\big\} \in P_2(H)$$

represents a menu that offers two recipes.

Finally, the third powerset

$$\mathcal{P}_3(H) = \mathcal{P}\big(\mathcal{P}_2(H)\big)$$

is the set of all possible "meal plans," i.e. collections of menus—useful, for instance, when planning multi-day catering events.

In general, the n-th powerset $\mathcal{P}_n(H)$ models n-level hierarchical groupings in everyday planning tasks.

Definition 0.5 (Fuzzy set)**.** [1,43] A *fuzzy set* τ in a non-empty universe Y is a mapping $\tau : Y \to [0,1]$. A *fuzzy relation* on Y is a fuzzy subset δ in $Y \times Y$. If τ is a fuzzy set in Y and δ is a fuzzy relation on Y, then δ is called a *fuzzy relation on* τ if

$$\delta(y,z) \leq \min\{\tau(y), \tau(z)\} \quad \text{for all } y, z \in Y.$$

Definition 0.6 (Hyperfuzzy Set)**.** [34–37,44] Let X be a non-empty universe. A *hyperfuzzy set* $\tilde{A}$ on X is defined by a mapping

$$\tilde{\mu} : X \longrightarrow \tilde{P}([0,1]),$$

where $\tilde{P}([0,1])$ denotes the collection of all non-empty subsets of the interval $[0,1]$.

For each element $x \in X$, $\tilde{\mu}(x) \subseteq [0,1]$ represents the *set of possible membership degrees* of x in the set $\tilde{A}$. This formulation allows for representing uncertainty or variability in the degree of membership, extending the classical fuzzy set (which assigns a single real number in $[0,1]$) to a set-valued interpretation.

Example 0.7 (Freshness Assessment via a Hyperfuzzy Set)**.** Let X be the set of ingredients in a simple salad:

$$X = \{\text{tomato, cheese, basil}\}.$$

Three quality-control experts each give a fuzzy score in $[0,1]$ for the "freshness" of each ingredient. We model their variability as a hyperfuzzy set $\tilde{F}$ on X, where

$$\tilde{\mu}(x) \ = \ \{\text{ scores provided by the three experts for } x\} \ \subseteq [0,1].$$

Concretely,

$$\tilde{\mu}(\text{tomato}) = \{0.7,\ 0.8,\ 0.9\}, \quad \tilde{\mu}(\text{cheese}) = \{0.6,\ 0.85,\ 0.9\}, \quad \tilde{\mu}(\text{basil}) = \{0.5,\ 0.75,\ 0.95\}.$$

Here each set $\tilde{\mu}(x)$ captures the range of expert judgments on how "fresh" the ingredient x is, allowing us to represent uncertainty and disagreement among evaluators.

Definition 0.8 (m,n-SuperHyperFuzzy Set)**.** [28,45,46] Let X be a nonempty set and let $m,n \in \mathbb{N}_0$. Define the nonempty k-th powerset of a set Y by

$$\mathcal{P}_0^*(Y) = Y, \quad \mathcal{P}_k^*(Y) = \mathcal{P}\big(\mathcal{P}_{k-1}^*(Y)\big) \setminus \{\emptyset\}, \quad k \geq 1.$$

In particular, $\mathcal{P}_m^*(X)$ is the family of all nonempty elements of the m-th iterated powerset of X, and $\mathcal{P}_n^*([0,1])$ is defined analogously. Then an (m,n)-SuperHyperFuzzy Set on X is a function

$$\tilde{\mu}_{m,n} : \mathcal{P}_m^*(X) \ \longrightarrow \ \tilde{\mathcal{P}}_n^*\big([0,1]\big), \quad A \ \mapsto \ \tilde{\mu}_{m,n}(A),$$

where $\tilde{\mathcal{P}}_n^*([0,1])$ denotes the collection of all nonempty subsets of $\mathcal{P}_n([0,1])$. Thus each $A \in \mathcal{P}_m^*(X)$ is assigned a nonempty family of membership-degree sets $\tilde{\mu}_{m,n}(A) \subseteq \mathcal{P}_n([0,1])$, capturing hierarchical uncertainty across both the m- and n-levels.

Example 0.9 (Project Team Performance via a $(2,2)$–SuperHyperFuzzy Set)**.** Let

$$X = \{\text{Yutaka, Masafumi, Shintaro}\}$$

, and take $m = 2$, $n = 2$. Then

$$\mathcal{P}_2^*(X) = \mathcal{P}\big(\mathcal{P}^*(X)\big) \setminus \{\emptyset\},$$

so one representative element is

$$A \ = \ \big\{\{\text{Yutaka, Masafumi}\},\ \{\text{Masafumi, Shintaro}\}\big\} \ \in \ \mathcal{P}_2^*(X).$$

We assign to A a nonempty family of second-level membership sets by

$$\tilde{\mu}_{2,2}(A) = \big\{\underbrace{\{\{0.7,\ 0.75\},\ \{0.8\}\}}_{L_1}, \underbrace{\{\{0.6\},\ \{0.85,\ 0.90\}\}}_{L_2}\big\} \ \subseteq \ \tilde{\mathcal{P}}_2^*([0,1]).$$

Here:

- Each inner set, for example $\{0.7, 0.75\}$, consists of possible membership scores from one evaluation method (e.g. quarterly metrics).
- The outer set $\{L_1, L_2\}$ captures two distinct evaluation scenarios (e.g. peer review vs. manager review).

Thus this $(2, 2)$–SuperHyperFuzzy Set models hierarchical uncertainty both in the team structure (level $m = 2$) and in the membership-assessment process (level $n = 2$).

Example 0.10 (Supply Chain Reliability Assessment via a $(2, 1)$–SuperHyperFuzzy Set)**.** Let $X = \{\mathrm{A, B, C}\}$ be a set of suppliers, and take $m = 2$, $n = 1$. Then

$$\mathcal{P}_1^*(X) = \big\{\{A\}, \{B\}, \{C\}, \{A, B\}, \{A, C\}, \{B, C\}, \{A, B, C\}\big\},$$

and

$$\mathcal{P}_2^*(X) = \mathcal{P}\big(\mathcal{P}_1^*(X)\big) \setminus \{\emptyset\},$$

so one representative element is

$$A \;=\; \big\{\{A, B\},\ \{B, C\}\big\} \;\in\; \mathcal{P}_2^*(X),$$

which models two overlapping supply-chain partnerships. We assign to A a set of possible reliability scores by

$$\tilde{\mu}_{2,1}(A) = \{\,0.75,\ 0.85,\ 0.92\} \;\subseteq\; \tilde{\mathcal{P}}_1^*([0, 1]) = \big\{\text{all nonempty subsets of } [0, 1]\big\}.$$

Here each value represents the aggregate reliability (on-time delivery, defect-rate compliance, audit score, etc.) under a different evaluation scenario. Thus this $(2, 1)$–SuperHyperFuzzy Set captures hierarchical uncertainty in the supplier network structure (level $m = 2$) while allowing multiple possible reliability judgments (level $n = 1$).

1.2|Fuzzy Linear Programming

Linear programming is an optimization technique that aims to maximize or minimize a linear objective function subject to a set of linear equality and inequality constraints(cf. [47–52]). Fuzzy linear programming generalizes this framework by incorporating fuzzy numbers into the objective coefficients and constraints, allowing for uncertainty in both parameters and feasible regions [53–57]. The definition of the Fuzzy Linear Programming is provided below.

Definition 0.11 (Fuzzy Linear Program)**.** (cf. [53]) Let $x = (x_1, \ldots, x_n)^\top \in \mathrm{R}^n_{\geq 0}$ be the decision vector. A *fuzzy linear program* (FLP) is specified by fuzzy numbers

$$\tilde{C}_j \;\; (j = 1, \ldots, n), \quad \tilde{A}_{ij} \;\; (i = 1, \ldots, m,\; j = 1, \ldots, n), \quad \tilde{B}_i \;\; (i = 1, \ldots, m),$$

and written as

$$\tilde{Z}(x) = \bigoplus_{j=1}^{n} \tilde{C}_j\, x_j \;\longrightarrow\; \max_{x\geq 0}, \tag{1}$$

$$\text{subject to} \quad \bigoplus_{j=1}^{n} \tilde{A}_{ij}\, x_j \;\preceq\; \tilde{B}_i, \quad i = 1, \ldots, m. \tag{2}$$

Here:

- "$\oplus$" denotes the *extended addition* of fuzzy numbers via Zadeh's extension principle:
$$\mu_{U\oplus V}(z) = \sup_{u+v=z} \min\{\mu_U(u),\ \mu_V(v)\}.$$
- "$\preceq$" is a chosen *fuzzy inequality* relation between fuzzy sets (e.g. the pessimistic "$<R$" of Zimmermann).
- Each fuzzy number $\tilde{C}_j$ (resp. $\tilde{A}_{ij}, \tilde{B}_i$) is a convex normalized fuzzy set on R with membership $\mu_{\tilde{C}_j}(c)$.

Example 0.12 (Fuzzy Production Planning)**.** (cf. [58, 59]) A factory produces two products $x_1, x_2 \geq 0$. Profit per unit is only approximately known:

$$\tilde{C}_1 = [5, 6]_{\mathrm{tr}}, \quad \tilde{C}_2 = [3, 4]_{\mathrm{tr}},$$

where $[a, b]_{\mathrm{tr}}$ denotes the triangular fuzzy number with support $[a, b]$ and peak at the midpoint. There is a single resource constraint: the available capacity is vague,

$$\tilde{B} = [\,90, 110\,]_{\mathrm{tr}},$$

and each unit of x_1 consumes 2–3 units, of x_2 consumes 1–2 units:

$$\tilde{A}_{11} = [2, 3]_{\mathrm{tr}}, \quad \tilde{A}_{12} = [1, 2]_{\mathrm{tr}}.$$

Thus the FLP is

$$\tilde{Z}(x) = \tilde{C}_1\, x_1 \oplus \tilde{C}_2\, x_2 \;\longrightarrow\; \max, \quad \tilde{A}_{11}\, x_1 \oplus \tilde{A}_{12}\, x_2 \;\preceq\; \tilde{B}, \quad x_1, x_2 \geq 0.$$

Concretely, under the usual min–+ extension, the objective's bounds are

$$\underline{Z}(x) = 5x_1 + 3x_2, \quad \overline{Z}(x) = 6x_1 + 4x_2,$$

and the resource constraint becomes the fuzzy interval

$$[\,2x_1 + 1x_2\,,\, 3x_1 + 2x_2\,] \;\preceq\; [\,90, 110\,].$$

One may solve by introducing a crisp satisfaction level $k \in [0, 1]$ and forming the corresponding parametric LP.

2|Results of This Papers

This section outlines the main results presented in this paper.

2.1|HyperFuzzy Linear Programming

The definition of the HyperFuzzy Linear Programming is provided below.

Definition 0.13 (Hyperfuzzy Number)**.** A *hyperfuzzy number* on R is a hyperfuzzy set

$$\tilde{C} : \mathrm{R} \;\longrightarrow\; \tilde{\mathcal{P}}([0, 1]),$$

such that each α-cut

$$\tilde{C}_\alpha = \{\, c \in \mathrm{R} \mid \sup \tilde{C}(c) \geq \alpha\}$$

is nonempty and compact. Hyperfuzzy numbers generalize classical fuzzy numbers by assigning to each real c a nonempty compact subset $\tilde{C}(c) \subseteq [0, 1]$ of possible membership degrees.

Definition 0.14 (Hyperaddition and Hyperscalar-Multiplication)**.** Let $\tilde{U}, \tilde{V}$ be hyperfuzzy numbers, and $\lambda \geq 0$ a scalar. Define:

$$(\tilde{U} \oplus_h \tilde{V})(z) = \bigcup_{u+v=z} \big(\tilde{U}(u) \,\cap\, \tilde{V}(v)\big),$$

$$(\lambda \otimes_h \tilde{U})(z) = \bigcup_{u=\frac{z}{\lambda}} \tilde{U}(u),$$

interpreting the unions over all decompositions of z. These extend the usual Zadeh extension principle to the hyperfuzzy setting.

Definition 0.15 (HyperLinear Program)**.** Let $x = (x_1, \ldots, x_n)^\top \in \mathrm{R}^n_{\geq 0}$ be the decision vector. A *HyperLinear Program* (HLP) is specified by hyperfuzzy numbers

$$\tilde{C}_j \;(j = 1, \ldots, n), \quad \tilde{A}_{ij} \;(i = 1, \ldots, m,\; j = 1, \ldots, n), \quad \tilde{B}_i \;(i = 1, \ldots, m),$$

and written as

$$\tilde{Z}(x) \;=\; \bigoplus_{j=1}^{n} x_j \otimes_h \tilde{C}_j \;\longrightarrow\; {}_{x \geq 0}, \tag{3}$$

$$\text{subject to} \quad \bigoplus_{j=1}^{n} x_j \otimes_h \tilde{A}_{ij} \;\preceq_h\; \tilde{B}_i, \quad i = 1, \ldots, m, \tag{4}$$

where "$\preceq_h$" is a chosen hyperfuzzy inequality (e.g. comparing α-cuts).

Example 0.16 (HyperLinear Production Planning)**.** A factory produces two products $x_1, x_2 \geq 0$. The profit coefficients are uncertain and modeled as hyperfuzzy numbers:

$$\tilde{C}_1(c) = \begin{cases} \{0.7, 0.8\}, & c = 5, \\ \{0.9\}, & c = 6, \\ \emptyset, & \text{otherwise}, \end{cases} \qquad \tilde{C}_2(c) = \begin{cases} \{0.6\}, & c = 3, \\ \{0.7, 0.75\}, & c = 4, \\ \emptyset, & \text{otherwise}. \end{cases}$$

Resource consumption per unit is also hyperfuzzy:

$$\tilde{A}_{11}(a) = \begin{cases} \{0.8\}, & a = 2, \\ \{0.9\}, & a = 3, \\ \emptyset, & \text{otherwise}, \end{cases} \qquad \tilde{A}_{12}(a) = \begin{cases} \{0.65\}, & a = 1, \\ \{0.85\}, & a = 2, \\ \emptyset, & \text{otherwise}, \end{cases}$$

and the available capacity is hyperfuzzy:

$$\tilde{B}(b) = \begin{cases} \{1.0\}, & b = 90, \\ \{0.5\}, & b = 110, \\ \emptyset, & \text{otherwise}. \end{cases}$$

Thus the HyperLinear Program is

$$\tilde{Z}(x) = x_1 \otimes_h \tilde{C}_1 \ \oplus_h \ x_2 \otimes_h \tilde{C}_2 \ \longrightarrow \ \max, \quad x_1 \otimes_h \tilde{A}_{11} \ \oplus_h \ x_2 \otimes_h \tilde{A}_{12} \ \preceq_h \ \tilde{B}, \ x_1, x_2 \geq 0.$$

Objective evaluation at $x = (10, 5)$**.** Possible profit values and membership degrees:

$$\begin{aligned} z = 5 \cdot 10 + 3 \cdot 5 = 65 : &\quad \mu = \min\{\sup \tilde{C}_1(5), \ \sup \tilde{C}_2(3)\} = \min\{0.8, \ 0.6\} = 0.6, \\ z = 5 \cdot 10 + 4 \cdot 5 = 70 : &\quad \mu = \min\{0.8, \ 0.75\} = 0.75, \\ z = 6 \cdot 10 + 3 \cdot 5 = 75 : &\quad \mu = \min\{0.9, \ 0.6\} = 0.6, \\ z = 6 \cdot 10 + 4 \cdot 5 = 80 : &\quad \mu = \min\{0.9, \ 0.75\} = 0.75. \end{aligned}$$

Hence $\tilde{Z}(10, 5)$ assigns $\{65 \mapsto 0.6, \ 70 \mapsto 0.75, \ 75 \mapsto 0.6, \ 80 \mapsto 0.75\}$.
Constraint evaluation at $x = (10, 5)$**.** Possible consumptions and memberships:

$$\begin{aligned} c = 2 \cdot 10 + 1 \cdot 5 = 25 : &\quad \mu = \min\{\sup \tilde{A}_{11}(2), \ \sup \tilde{A}_{12}(1)\} = \min\{0.8, \ 0.65\} = 0.65, \\ c = 2 \cdot 10 + 2 \cdot 5 = 30 : &\quad \mu = \min\{0.8, \ 0.85\} = 0.8, \\ c = 3 \cdot 10 + 1 \cdot 5 = 35 : &\quad \mu = \min\{0.9, \ 0.65\} = 0.65, \\ c = 3 \cdot 10 + 2 \cdot 5 = 40 : &\quad \mu = \min\{0.9, \ 0.85\} = 0.85. \end{aligned}$$

Comparing to capacity:

$$\sup \tilde{B}(90) = 1.0, \quad \sup \tilde{B}(110) = 0.5.$$

All consumptions $c \leq 90$ yield satisfaction $\min\{\mu, 1.0\} = \mu$, so $x = (10, 5)$ is feasible with minimum satisfaction 0.65.

Thus this example demonstrates a HyperLinear Program's evaluation under multiple profit and resource scenarios, with membership degrees quantified by the hyperfuzzy arithmetic $\otimes_h, \oplus_h$.

Theorem 0.17 (Generalization of Linear Programming)**.** *Every classical linear program is a special case of an HLP in which all hyperfuzzy numbers* $\tilde{C}_j, \tilde{A}_{ij}, \tilde{B}_i$ *have singleton-valued membership sets. Conversely, if in an HLP all* $\tilde{C}_j, \tilde{A}_{ij}, \tilde{B}_i$ *are hyperfuzzy numbers with point-singleton membership at each real argument, then the HLP reduces exactly to the corresponding classical LP.*

Proof: **(1) LP $\implies$ HLP.** Given a classical LP with crisp coefficients c_j, A_{ij}, B_i, define hyperfuzzy numbers by

$$\tilde{C}_j(c) = \begin{cases} \{1\}, & c = c_j, \\ \emptyset, & \text{otherwise}, \end{cases} \qquad \tilde{A}_{ij}(a) = \begin{cases} \{1\}, & a = A_{ij}, \\ \emptyset, & \text{otherwise}, \end{cases} \qquad \tilde{B}_i(b) = \begin{cases} \{1\}, & b = B_i, \\ \emptyset, & \text{otherwise}. \end{cases}$$

Then for any x, $\bigoplus_j x_j \otimes_h \tilde{C}_j$ has support exactly at the single scalar $\sum_j c_j x_j$, and similarly each constraint reproduces $\sum_j A_{ij} x_j \leq B_i$. Thus the HLP (3)–(4) coincides with the original LP.

(2) HLP with singletons $\implies$ LP. Conversely, if each hyperfuzzy number $\tilde{C}_j$ satisfies $\tilde{C}_j(c) \subseteq \{1\}$ or $\emptyset$, then it encodes a unique crisp coefficient c_j. The hyperaddition $\oplus_h$ and hyperscalar-multiplication $\otimes_h$ on singleton-valued hyperfuzzy sets reduce to ordinary addition and multiplication. Hence the HLP reduces to the classical LP with coefficients c_j, A_{ij}, B_i.

(3) Hyperfuzzy Structure. By Definitions 0.15 and of hyperfuzzy numbers, each coefficient mapping takes values in $\tilde{\mathcal{P}}([0, 1])$, endowing the objective and constraint functions with a hyperfuzzy set structure. Therefore HLPs both generalize LPs and inherit the algebraic richness of hyperfuzzy set theory. □

2.2|(m,n)-SuperHyperfuzzy Programming

The definition of the (m,n)-SuperHyperfuzzy Programming is provided below.

Definition 0.18 ((m,n)-SuperHyperfuzzy Number)**.** Let $m,n \geq 0$. Define recursively the nonempty k-th powerset of R by

$$\mathcal{P}_0^*(\mathrm{R}) = \mathrm{R}, \quad \mathcal{P}_k^*(\mathrm{R}) = \mathcal{P}\big(\mathcal{P}_{k-1}^*(\mathrm{R})\big) \setminus \{\emptyset\}, \quad k \geq 1.$$

An (m,n)-*superhyperfuzzy number* is a mapping

$$\tilde{U} \colon \mathcal{P}_m^*(\mathrm{R}) \longrightarrow \tilde{\mathcal{P}}_n^*([0,1]),$$

such that for each $A \in \mathcal{P}_m^*(\mathrm{R})$, $\tilde{U}(A) \subseteq \mathcal{P}_n([0,1])$ is nonempty and compact, and for each $\alpha \in (0,1]$ the α-cut

$$\tilde{U}_\alpha = \big\{ A \in \mathcal{P}_m^*(\mathrm{R}) \mid \sup^{(n)} \tilde{U}(A) \geq \alpha \big\}$$

is a nonempty compact subset of $\mathcal{P}_m^*(\mathrm{R})$. Here $\sup^{(n)}$ denotes the n-fold supremum in $\tilde{\mathcal{P}}_n^*([0,1])$.

Definition 0.19 ((m,n)-SuperHyperaddition and (m,n)-SuperHyperscalar-Multiplication)**.** Let $\tilde{U}, \tilde{V}$ be (m,n)-superhyperfuzzy numbers and $\lambda \geq 0$. For $C \in \mathcal{P}_m^*(\mathrm{R})$, define

$$(\tilde{U} \oplus_{m,n} \tilde{V})(C) = \bigcup_{\substack{X,Y \in \mathcal{P}_m^*(\mathrm{R}) \\ X+Y=C}} \big(\tilde{U}(X) \otimes_{m,n} \tilde{V}(Y)\big), \quad X+Y = \{\, x+y \mid x \in X,\ y \in Y \},$$

and

$$(\lambda \otimes_{m,n} \tilde{U})(C) = \bigcup_{\substack{X \in \mathcal{P}_m^*(\mathrm{R}) \\ \lambda X = C}} \tilde{U}(X), \quad \lambda X = \{\lambda x \mid x \in X\}.$$

Here the (m,n)-superhyperproduct $\otimes_{m,n}$ is given recursively by

$$A \otimes_{m,1} B = \{\, a \cdot b \mid a \in A,\ b \in B \}, \quad A \otimes_{m,k} B = \bigcup_{X \in A,\, Y \in B} \big(X \otimes_{m,k-1} Y\big), \quad k \geq 2.$$

Definition 0.20 ((m,n)-SuperHyperFuzzy Linear Program)**.** Let $x = (x_1, \ldots, x_N)^\top \in \mathrm{R}_{\geq 0}^N$. An (m,n)-*superhyperlinear program* is specified by (m,n)-superhyperfuzzy numbers $\tilde{C}_j, \tilde{A}_{ij}, \tilde{B}_i$ for $j = 1, \ldots, N$, $i = 1, \ldots, M$, and takes the form

$$\tilde{Z}(x) = \bigoplus_{m,n\,;j=1}^{N} x_j \otimes_{m,n} \tilde{C}_j \longrightarrow {}_{x \geq 0}, \tag{5}$$

$$\text{s.t.} \quad \bigoplus_{m,n\,;j=1}^{N} x_j \otimes_{m,n} \tilde{A}_{ij} \preceq_{m,n} \tilde{B}_i, \quad i = 1, \ldots, M, \tag{6}$$

where $\preceq_{m,n}$ is a chosen ordering on $\tilde{\mathcal{P}}_n^*([0,1])$ (for instance, comparing all level–α cuts).

Example 0.21 (A $(1,2)$-SuperHyperFuzzy Linear Production Planning)**.** Consider a factory producing two products $x_1, x_2 \geq 0$. We model profit and resource data as $(1,2)$-superhyperfuzzy numbers $\tilde{C}_j : \mathcal{P}_1^*(\mathrm{R}) = \mathcal{P}(\mathrm{R}) \setminus \{\emptyset\} \to \tilde{\mathcal{P}}_2^*([0,1])$. For example:

$$\tilde{C}_1(\{5\}) = \big\{\{\{0.7, 0.75\}, \{0.8\}\}, \{\{0.9\}\}\big\}, \quad \tilde{C}_1(\{6\}) = \big\{\{\{0.85\}\}\big\},$$

and similarly for $\tilde{C}_2$. Resource consumption $\tilde{A}_{ij}$ and capacity $\tilde{B}_i$ are defined analogously.

Then the $(1,2)$-SHLP is

$$\tilde{Z}(x) = x_1 \otimes_{1,2} \tilde{C}_1 \oplus_{1,2} x_2 \otimes_{1,2} \tilde{C}_2 \longrightarrow \max, \quad x_1 \otimes_{1,2} \tilde{A}_{11} \oplus_{1,2} x_2 \otimes_{1,2} \tilde{A}_{12} \preceq_{1,2} \tilde{B}, \quad x_1, x_2 \geq 0.$$

Evaluating at $x = (10, 5)$ produces a nested family of possible profit values and satisfaction levels, reflecting one level of set-valued uncertainty in the coefficients ($m = 1$) and two levels in the membership degrees ($n = 2$).

Example 0.22 (A $(0,3)$-SuperHyperFuzzy Linear Production Planning)**.** Consider a factory producing two products $x_1, x_2 \geq 0$. We model both profit and resource coefficients as $(0,3)$-superhyperfuzzy numbers $\tilde{C}_j : \mathcal{P}_0^*(\mathrm{R}) = \mathrm{R} \to \tilde{\mathcal{P}}_3^*([0,1])$:

$$\tilde{C}_1(c) = \begin{cases} \big\{\underbrace{\{\{0.6, 0.65\}, \{0.7\}\}}_{L_1}, \underbrace{\{\{0.75\}, \{0.8, 0.85\}\}}_{L_2}\big\}, & c = 5, \\ \{\{\{0.9\}\}\}, & c = 6, \\ \emptyset, & \text{otherwise}, \end{cases}$$

$$\tilde{C}_2(c) = \begin{cases} \{\underbrace{\{\{0.5\},\ \{0.55, 0.6\}\}}_{M_1},\ \underbrace{\{\{0.65\},\ \{0.7, 0.75\}\}}_{M_2}\}, & c = 3, \\ \{\{\{0.8\}\}\}, & c = 4, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Resource consumption per unit is similarly given by $(0, 3)$-superhyperfuzzy numbers $\tilde{A}_{ij} : \mathrm{R} \to \tilde{\mathcal{P}}_3^*([0, 1])$, for instance

$$\tilde{A}_{11}(a) = \begin{cases} \{\{\{0.7\}\},\ \{\{0.8\}, \{0.85\}\}\}, & a = 2, \\ \{\{\{0.9\}\}\}, & a = 3, \\ \emptyset, & \text{otherwise,} \end{cases} \qquad \tilde{A}_{12}(a) = \begin{cases} \{\{\{0.6\},\ \{0.65\}\}\}, & a = 1, \\ \{\{\{0.75\},\ \{0.8\}\}\}, & a = 2, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Available capacity $\tilde{B} : \mathrm{R} \to \tilde{\mathcal{P}}_3^*([0, 1])$ is defined by

$$\tilde{B}(b) = \begin{cases} \{\{\{1.0\}\}\}, & b = 100, \\ \{\{\{0.5\}\}\}, & b = 120, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Then the $(0, 3)$-SHLP takes the form

$$\tilde{Z}(x) = x_1 \otimes_{0,3} \tilde{C}_1 \ \oplus_{0,3}\ x_2 \otimes_{0,3} \tilde{C}_2 \ \longrightarrow_{x \geq 0}, \quad x_1 \otimes_{0,3} \tilde{A}_{11} \ \oplus_{0,3}\ x_2 \otimes_{0,3} \tilde{A}_{12} \ \preceq_{0,3}\ \tilde{B}, \quad x_1, x_2 \geq 0.$$

Objective evaluation at $x = (10, 5)$**.** Compute

$$10 \otimes_{0,3} \tilde{C}_1 = \{\underbrace{\{\{6, 6.5\},\ \{7\}\}}_{10L_1}, \underbrace{\{\{7.5\},\ \{8, 8.5\}\}}_{10L_2}\}, \quad 5 \otimes_{0,3} \tilde{C}_2 = \{\{\{2.5\},\ \{2.75, 3\}\}, \{\{4\}\}\}.$$

Their $(0, 3)$-superhyper-sum yields four nested sets—for example $\{\{6, 6.5\}, \{7\}\} \cup \{\{2.5\}, \{2.75, 3\}\}$—capturing all combined profit possibilities.

Constraint evaluation at $x = (10, 5)$**.** Similarly,

$$10 \otimes_{0,3} \tilde{A}_{11} = \{\{\{7\}\},\ \{\{8\}, \{8.5\}\}\}, \quad 5 \otimes_{0,3} \tilde{A}_{12} = \{\{\{3\}, \{3.25\}\}\}.$$

Their $(0, 3)$-superhyper-sum is $\{\{7, 3\}, \{7, 3.25\}, \{8, 3\}, \{8, 3.25\}, \{8.5, 3\}, \{8.5, 3.25\}\}$. Since $\sup^{(3)} 100 = 1.0$ and $\sup^{(3)} 120 = 0.5$, all consumption sets satisfy the capacity constraint at full membership.

This example shows how a $(0, 3)$-SHLP captures three nested levels of uncertainty in both objective and constraints, producing a hierarchy of potential outcomes and feasibility degrees.

Example 0.23 (A $(2, 2)$-SuperHyperFuzzy Linear Production Planning)**.** Consider a factory that produces two products $x_1, x_2 \geq 0$. We model both profit coefficients and resource-consumption coefficients as $(2, 2)$–superhyperfuzzy numbers:

$$\tilde{C}_1(c) = \begin{cases} \{\underbrace{\{\{0.6, 0.65\},\ \{0.7\}\}}_{L_1},\ \underbrace{\{\{0.8\}\}}_{L_2}\}, & c = 10, \\ \{\underbrace{\{\{0.9, 0.95\}\}}_{L_3},\ \underbrace{\{\{0.85\}\}}_{L_4}\}, & c = 12, \\ \emptyset, & \text{otherwise,} \end{cases}$$

$$\tilde{C}_2(c) = \begin{cases} \{\underbrace{\{\{0.5, 0.55\},\ \{0.6\}\}}_{M_1},\ \underbrace{\{\{0.65\},\ \{0.7, 0.75\}\}}_{M_2}\}, & c = 8, \\ \{\underbrace{\{\{0.8\}\}}_{M_3},\ \underbrace{\{\{0.85, 0.9\}\}}_{M_4}\}, & c = 6, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Resource consumption per unit is similarly given by

$$\tilde{A}_{11}(a)=\begin{cases}\big\{\underbrace{\{\{0.7\},\ \{0.75\}\}}_{R_1},\ \underbrace{\{\{0.8\}\}}_{R_2}\big\}, & a=4,\\ \big\{\underbrace{\{\{0.85\}\}}_{R_3},\ \underbrace{\{\{0.9,0.95\}\}}_{R_4}\big\}, & a=5,\\ \emptyset, & \text{otherwise,}\end{cases}\qquad \tilde{A}_{12}(a)=\begin{cases}\big\{\underbrace{\{\{0.6\},\ \{0.65,0.7\}\}}_{S_1}\big\}, & a=3,\\ \big\{\underbrace{\{\{0.8\},\ \{0.85\}\}}_{S_2}\big\}, & a=4,\\ \emptyset, & \text{otherwise.}\end{cases}$$

The available capacity is

$$\tilde{B}(b)=\begin{cases}\big\{\underbrace{\{\{1.0\}\}}_{T_1},\ \underbrace{\{\{0.5\}\}}_{T_2}\big\}, & b=100,\\ \big\{\underbrace{\{\{0.9\},\ \{0.95\}\}}_{T_3}\big\}, & b=120,\\ \emptyset, & \text{otherwise.}\end{cases}$$

Then the $(2,2)$-SuperHyperLinear Program reads

$$\tilde{Z}(x)=x_1\otimes_{2,2}\tilde{C}_1\ \oplus_{2,2}\ x_2\otimes_{2,2}\tilde{C}_2\ \longrightarrow\ {}_{x\geq 0},$$

$$\text{s.t.}\quad x_1\otimes_{2,2}\tilde{A}_{11}\ \oplus_{2,2}\ x_2\otimes_{2,2}\tilde{A}_{12}\ \preceq_{2,2}\ \tilde{B},\quad x_1,x_2\geq 0.$$

Objective evaluation at $(x_1,x_2)=(2,3)$.

$$2\otimes_{2,2}\tilde{C}_1:\begin{cases}20\mapsto\{L_1,L_2\},\\ 24\mapsto\{L_3,L_4\},\end{cases}\qquad 3\otimes_{2,2}\tilde{C}_2:\begin{cases}24\mapsto\{M_1,M_2\},\\ 18\mapsto\{M_3,M_4\}.\end{cases}$$

Thus there are four profit scenarios:

$$\tilde{Z}(2,3)=\begin{cases}44: & \{\,L_1\cup M_1,\ L_1\cup M_2,\ L_2\cup M_1,\ L_2\cup M_2\,\},\\ 38: & \{\,L_1\cup M_3,\ L_1\cup M_4,\ L_2\cup M_3,\ L_2\cup M_4\,\},\\ 48: & \{\,L_3\cup M_1,\ L_3\cup M_2,\ L_4\cup M_1,\ L_4\cup M_2\,\},\\ 42: & \{\,L_3\cup M_3,\ L_3\cup M_4,\ L_4\cup M_3,\ L_4\cup M_4\,\}.\end{cases}$$

Constraint evaluation at $(x_1,x_2)=(2,3)$.

$$2\otimes_{2,2}\tilde{A}_{11}:\begin{cases}8\mapsto\{R_1,R_2\},\\ 10\mapsto\{R_3,R_4\},\end{cases}\qquad 3\otimes_{2,2}\tilde{A}_{12}:\begin{cases}9\mapsto\{S_1\},\\ 12\mapsto\{S_2\}.\end{cases}$$

Their superhyper-sum yields four consumption scenarios:

$$17:\{R_1\cup S_1,\ R_2\cup S_1\},\quad 18:\{R_1\cup S_2,\ R_2\cup S_2\},$$
$$19:\{R_3\cup S_1,\ R_4\cup S_1\},\quad 22:\{R_3\cup S_2,\ R_4\cup S_2\}.$$

Comparing to capacity $\tilde{B}$ (at 100 and 120), all consumptions ≤ 100 satisfy the constraint with full membership, so $(2,3)$ is feasible at degree 1.

This example demonstrates how a $(2,2)$-SHLP captures two nested levels of set-valued uncertainty in both the objective and the constraints, producing a rich hierarchy of possible outcomes and satisfaction degrees.

Example 0.24 (A $(1,3)$-SuperHyperFuzzy Linear Production Planning)**.** Consider a factory producing two products $x_1,x_2\geq 0$. We model both profit coefficients and resource-consumption coefficients as $(1,3)$–superhyperfuzzy numbers

$$\tilde{C}_j:\mathcal{P}_1^*(\mathbb{R})\to\tilde{\mathcal{P}}_3^*([0,1])$$

and

$$\tilde{A}_{ij}:\mathcal{P}_1^*(\mathbb{R})\to\tilde{\mathcal{P}}_3^*([0,1])$$

. For example:

$$\tilde{C}_1(c) = \begin{cases} \big\{\underbrace{\{\{0.7, 0.75\}, \{0.8\}\}}_{L_1}, \underbrace{\{\{0.85\}, \{0.9, 0.95\}\}}_{L_2}\big\}, & c = 10, \\ \{\{\{0.9\}\}\}, & c = 12, \\ \emptyset, & \text{otherwise}, \end{cases}$$

$$\tilde{C}_2(c) = \begin{cases} \big\{\underbrace{\{\{0.5\}, \{0.55, 0.6\}\}}_{M_1}, \underbrace{\{\{0.65\}, \{0.7, 0.75\}\}}_{M_2}\big\}, & c = 8, \\ \{\{\{0.8\}\}\}, & c = 6, \\ \emptyset, & \text{otherwise}. \end{cases}$$

Similarly, resource consumption per unit is

$$\tilde{A}_{11}(a) = \begin{cases} \big\{\underbrace{\{\{0.7\}, \{0.75\}\}}_{R_1}, \underbrace{\{\{0.8\}, \{0.85\}\}}_{R_2}\big\}, & a = 4, \\ \{\{\{0.9\}\}\}, & a = 5, \\ \emptyset, & \text{otherwise}, \end{cases}$$

$$\tilde{A}_{12}(a) = \begin{cases} \big\{\underbrace{\{\{0.6\}, \{0.65, 0.7\}\}}_{S_1}\big\}, & a = 3, \\ \{\{\{0.95\}\}\}, & a = 4, \\ \emptyset, & \text{otherwise}. \end{cases}$$

The available capacity is

$$\tilde{B}(b) = \begin{cases} \{\{\{1.0\}\}\}, & b = 100, \\ \{\{\{0.5\}\}\}, & b = 120, \\ \emptyset, & \text{otherwise}. \end{cases}$$

Then the $(1, 3)$–SuperHyperLinear Program takes the form

$$\tilde{Z}(x) = x_1 \otimes_{1,3} \tilde{C}_1 \ \oplus_{1,3} \ x_2 \otimes_{1,3} \tilde{C}_2 \ \longrightarrow \ _{x \geq 0},$$

$$\text{subject to} \quad x_1 \otimes_{1,3} \tilde{A}_{11} \ \oplus_{1,3} \ x_2 \otimes_{1,3} \tilde{A}_{12} \ \preceq_{1,3} \ \tilde{B}, \quad x_1, x_2 \geq 0.$$

Objective evaluation at $(x_1, x_2) = (2, 3)$**.**

$$2 \otimes_{1,3} \tilde{C}_1 : \begin{cases} 20 \mapsto \{L_1, L_2\}, \\ 24 \mapsto \{\{\{0.9\}\}\}, \end{cases} \qquad 3 \otimes_{1,3} \tilde{C}_2 : \begin{cases} 24 \mapsto \{M_1, M_2\}, \\ 18 \mapsto \{\{\{0.8\}\}\}. \end{cases}$$

Hence the objective yields four profit-scenarios:

$$\tilde{Z}(2, 3) = \begin{cases} 44 : \ \{L_1 \cup M_1, \ L_1 \cup M_2, \ L_2 \cup M_1, \ L_2 \cup M_2\}, \\ 38 : \ \{L_1 \cup \{\{0.8\}\}, \ L_2 \cup \{\{0.8\}\}\}, \\ 48 : \ \{\{\{0.9\}\}\} \cup M_1, \ \{\{\{0.9\}\}\} \cup M_2, \\ 42 : \ \{\{\{0.9\}\}\} \cup \{\{0.8\}\}. \end{cases}$$

Constraint evaluation at $(x_1, x_2) = (2, 3)$**.**

$$2 \otimes_{1,3} \tilde{A}_{11} : \begin{cases} 8 \mapsto \{R_1, R_2\}, \\ 10 \mapsto \{\{\{0.9\}\}\}, \end{cases} \qquad 3 \otimes_{1,3} \tilde{A}_{12} : \begin{cases} 9 \mapsto \{S_1\}, \\ 12 \mapsto \{\{\{0.95\}\}\}. \end{cases}$$

Their superhyper-sum produces four consumption scenarii:

$$\begin{aligned} &17 : \{R_1 \cup S_1, \ R_2 \cup S_1\}, \\ &20 : \{R_1 \cup \{\{0.95\}\}, \ R_2 \cup \{\{0.95\}\}\}, \\ &19 : \{\{\{0.9\}\}\} \cup S_1, \\ &22 : \{\{\{0.9\}\}\} \cup \{\{0.95\}\}. \end{aligned}$$

Since all consumptions ≤ 100, each scenario satisfies the constraint at membership level drawn from $\tilde{B}(100) = \{\{1.0\}\}$. Thus $(2, 3)$ is feasible with full satisfaction.

This example illustrates how a $(1, 3)$-SuperHyperLinear Program captures one level of set-valued uncertainty in coefficients and three nested levels in membership degrees, yielding a rich scenario analysis for both objective and constraints.

Theorem 0.25 (Reduction of (m, n)-SuperHyperFuzzy Linear Programs)**.** *Let $m, n \geq 0$. An (m, n)-SuperHyperFuzzy Linear Program, as given in Definition 0.20, enjoys the following reductions:*

(i) ***Recovery of HyperFuzzy LP.*** *If we set $m = 0$ and $n = 1$, then*

$$\mathcal{P}_0^*(\mathrm{R}) = \mathrm{R},$$

$$\tilde{\mathcal{P}}_1^*([0, 1]) = \{ S \subseteq [0, 1] \mid S \neq \emptyset\},$$

and the superhyper-operations $\oplus_{0,1}$, $\otimes_{0,1}$, $\preceq_{0,1}$ reduce exactly to the hyperaddition, hyperscalar-multiplication, and fuzzy ordering of a HyperFuzzy Linear Program (Definition 0.15). Hence every HyperFuzzy LP is a special case of the (m, n)-SHLP.

(ii) ***Reduction to level-$n = 1$ in the codomain.*** *Suppose each coefficient map $\tilde{C}_j : \mathcal{P}_m^*(\mathrm{R}) \to \tilde{\mathcal{P}}_n^*([0, 1])$ actually takes values in the subfamily $\tilde{\mathcal{P}}_1^*([0, 1]) \subseteq \tilde{\mathcal{P}}_n^*([0, 1])$. Then all membership-degree operations $\oplus_{m,n}$, $\otimes_{m,n}$, $\preceq_{m,n}$ restrict to the level-1 versions $\oplus_{m,1}$, $\otimes_{m,1}$, $\preceq_{m,1}$, and the program reduces to an $(m, 1)$-SuperHyperLinear Program.*

(iii) ***Recovery of HyperFuzzy LP from real-valued domain.*** *If, in addition, each coefficient map's domain is $\mathcal{P}_m^*(\mathrm{R}) = \mathrm{R}$, then the coefficient domain collapses to real numbers and the program further reduces to a HyperFuzzy Linear Program, with no remaining set-valued or hierarchical structure.*

Proof: We verify each statement in turn.

(i) Recovery of HyperFuzzy LP. By choosing $m = 0$ and $n = 1$, the domain $\mathcal{P}_0^*(\mathrm{R})$ becomes R, and the codomain $\tilde{\mathcal{P}}_1^*([0, 1])$ is exactly all nonempty subsets of $[0, 1]$. Under these parameters, the superhyper-addition $\oplus_{0,1}$, superhyper-scalar multiplication $\otimes_{0,1}$, and ordering $\preceq_{0,1}$ coincide with the operations defined for a HyperFuzzy Linear Program (Definition 0.15). Thus every HyperFuzzy LP embed in the (m, n)-SHLP framework.

(ii) Reduction to level-1 codomain. If each $\tilde{C}_j$ factors through $\tilde{\mathcal{P}}_1^*([0, 1])$, then for all $A \in \mathcal{P}_m^*(\mathrm{R})$ we have $\tilde{C}_j(A) \subseteq \tilde{\mathcal{P}}_1^*([0, 1])$. By definition of the superhyper-operations, this condition forces $\oplus_{m,n}$, $\otimes_{m,n}$, and $\preceq_{m,n}$ to act as $\oplus_{m,1}$, $\otimes_{m,1}$, and $\preceq_{m,1}$ respectively. Hence the original (m, n)-SHLP is equivalent to an $(m, 1)$-SHLP.

(iii) Recovery of HyperFuzzy LP from real domain. Finally, if the domain also collapses via $\mathcal{P}_m^*(\mathrm{R}) = \mathrm{R}$, then all coefficient maps reduce to $\tilde{C}_j : \mathrm{R} \to \tilde{\mathcal{P}}_1^*([0, 1])$, and the program becomes precisely a HyperFuzzy Linear Program with operations $\oplus_{0,1}$, $\otimes_{0,1}$, and $\preceq_{0,1}$.

This completes the proof. □

Theorem 0.26 (Classical Linear Program as a Special Case)**.** *Let $m, n \geq 0$. Suppose in an (m, n)-SuperHyperFuzzy Linear Program (Definition 0.20) each coefficient mapping*

$$\tilde{C}_j : \mathcal{P}_m^*(\mathrm{R}) \longrightarrow \tilde{\mathcal{P}}_n^*([0, 1])$$

satisfies

$$\tilde{C}_j(A) = \{\{c_j\}\} \quad \textit{for some fixed } c_j \in \mathrm{R} \quad (\textit{and similarly for } \tilde{A}_{ij},\ \tilde{B}_i).$$

Then the program (5)–(6) *reduces exactly to the classical linear program*

$$\max_{x \geq 0} \sum_{j=1}^{N} c_j\, x_j, \quad \textit{s.t.} \sum_{j=1}^{N} A_{ij}\, x_j \leq B_i,\ i = 1, \ldots, M,$$

where A_{ij}, B_i are the crisp values arising from $\tilde{A}_{ij}$ and $\tilde{B}_i$.

Proof: Since each $\tilde{C}_j(A)$ is the singleton $\{c_j\}$, the superhyper-scalar multiplication $\otimes_{m,n}$ satisfies

$$x_j \otimes_{m,n} \tilde{C}_j = \{ x_j \cdot c_j \} \subset \mathrm{R},$$

and the superhyper-addition $\oplus_{m,n}$ of singletons reduces to ordinary addition. Likewise, the constraint operations collapse to $\sum_j A_{ij}x_j \le B_i$. Hence the (m,n)-SuperHyperLinear Program coincides with the classical LP. □

Theorem 0.27 (α–Cut Decomposition)**.** *For each $\alpha \in (0,1]$, define the α–cuts of the coefficient mappings by*

$$C_{j,\alpha} = \left\{ A \in \mathcal{P}_m^*(\mathrm{R}) \mid \sup^{(n)} \tilde{C}_j(A) \ge \alpha \right\},$$

$$A_{ij,\alpha} = \left\{ A \in \mathcal{P}_m^*(\mathrm{R}) \mid \sup^{(n)} \tilde{A}_{ij}(A) \ge \alpha \right\}, \quad B_{i,\alpha} = \left\{ b \in \mathrm{R} \mid \sup^{(n)} \tilde{B}_i(b) \ge \alpha \right\}.$$

Then the family of classical programs

$$\max_{x\ge 0} \max_{c_j \in C_{j,\alpha}} \sum_{j=1}^{N} c_j x_j, \quad \text{s.t.} \max_{a_{ij}\in A_{ij,\alpha}} \sum_{j=1}^{N} a_{ij} x_j \le \min_{b_i \in B_{i,\alpha}} b_i, \quad i = 1, \ldots, M,$$

parameterized by α, exactly describes the α–level behavior of the (m,n)-SuperHyperLinear Program. In particular:

- *The fuzzy feasible region of the (m,n)-SHLP is $\bigcap_{\alpha\in(0,1]}\{x \mid x \text{ feasible at level } \alpha\}$.*
- *The fuzzy objective $\tilde{Z}(x)$ is recovered by collecting, for each α, the optimal values of these crisp programs.*

Proof: By definition of the α–cut, $C_{j,\alpha}$ collects precisely those domain values A whose membership supremum is at least α. For any x, the value of $x_j \otimes_{m,n} \tilde{C}_j$ at level α is the set $\{x_j \cdot c \mid c \in C_{j,\alpha}\}$. Superhyper-addition of these sets then yields the interval $\left[\min_{c\in C_{j,\alpha}} \sum c\, x_j,\ \max_{c\in C_{j,\alpha}} \sum c\, x_j\right]$. Requiring this interval to lie below $\min B_{i,\alpha}$ enforces the crisp inequality $\max_{a\in A_{ij,\alpha}} \sum a\, x_j \le \min_{b\in B_{i,\alpha}} b$. Collecting over all α recovers the full fuzzy program. □

Theorem 0.28 (Monotonicity of α–Level Feasible Sets)**.** *Let $\alpha_1, \alpha_2 \in (0,1]$ with $\alpha_1 < \alpha_2$. Then*

$$C_{j,\alpha_2} \subseteq C_{j,\alpha_1}, \quad A_{ij,\alpha_2} \subseteq A_{ij,\alpha_1}, \quad B_{i,\alpha_2} \subseteq B_{i,\alpha_1},$$

and consequently the α_2–level feasible region is contained in the α_1–level feasible region.

Proof: By construction, if $\sup^{(n)} \tilde{C}_j(A) \ge \alpha_2 > \alpha_1$ then certainly $\sup^{(n)} \tilde{C}_j(A) \ge \alpha_1$. The same argument applies to $\tilde{A}_{ij}$ and $\tilde{B}_i$. Hence each higher–α cut is nested inside the lower–α cut, and the corresponding feasible regions satisfy the stated containment. □

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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