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Chaos-Preserving Fuzzy Difference Operators: A Bridge Between Fuzzy Arithmetic and Fuzzy Dynamical Systems

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Abstract

In this work, we propose a novel framework that links generalized fuzzy difference operators—defined through α-level set constructions—with the dynamical behavior of fuzzy systems. By revisiting the compatibility between fuzzy set operations and their α-level counterparts, we introduce the concept of chaos-preserving operators, i.e., binary fuzzy operations that maintain or amplify chaotic dynamics under the Zadeh extension. We demonstrate that, under specific structural conditions (such as upper semicontinuity and nestedness of level sets), certain generalized Hausdorff-type differences not only admit consistent fuzzy representations but also preserve Devaney chaos, Li–Yorke chaos, and distributional chaos in fuzzy dynamical systems. Our theoretical development is supported by explicit constructions involving triangular fuzzy numbers and set-valued dynamics. The proposed framework opens a new avenue for analyzing uncertainty-propagating chaos in fuzzy environments, with potential applications in nonlinear systems, decision theory, and complex modeling.

Keywords: Fuzzy dynamical systems, Chaos preservation, Zadeh extension, Hausdorff fuzzy difference, α-level sets, Li–Yorke chaos, Fuzzy arithmetic, Type-I and Type-II fuzzy difference, Set-valued dynamics, Topological compatibility.

1 | Introduction

The study of fuzzy dynamical systems has received increasing attention in recent years, driven by the need to model complex behaviors under uncertainty. The Zadeh extension of a continuous map $f: X \to X$ induces a fuzzy dynamical system on the space $\mathcal{F}(X)$ of normal, upper semicontinuous fuzzy sets with compact support.

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Understanding how classical dynamical properties—such as chaos, sensitivity, or transitivity—translate into this fuzzy setting has proven both challenging and insightful.

On a parallel track, the theory of fuzzy arithmetic has evolved to consider increasingly sophisticated notions of operations between fuzzy numbers and sets. In particular, recent work by [1] has laid a rigorous foundation for generalized fuzzy difference operators constructed via α-level sets, extending and correcting earlier approaches by [2], [3]. These developments ensure the existence and consistency of such operations under precise topological and structural assumptions.

Despite these advances, a fundamental question remains largely unexplored: How do fuzzy arithmetic operations—especially those involving differences—affect the dynamical behavior of fuzzy systems? Can certain operations preserve or even enhance chaotic behavior under the induced fuzzy dynamics?

This paper addresses these questions by proposing a unified framework that connects fuzzy difference operators with chaotic dynamics in $\mathcal{F}(X)$. Specifically, we introduce the concept of chaos-preserving fuzzy difference operators, and we show that several classes of generalized differences, including type-I, type-II [4], and Hausdorff-type differences, preserve key features of chaos when applied within Zadeh-extended systems. Furthermore, we establish new results that link the structural compatibility of α-level sets with the preservation of chaos in fuzzy dynamics.

Main Contributions

We propose a new class of operators, referred to as chaos-preserving fuzzy difference operators, which maintain key dynamical properties under the Zadeh extension.

We establish existence and consistency theorems for type-I, type-II, and Hausdorff fuzzy differences when applied in dynamic contexts.

We provide sufficient conditions under which these operations preserve topological transitivity, density of periodic points, and various forms of chaos.

We illustrate the theoretical results with explicit numerical examples, using triangular fuzzy numbers and classical chaotic maps such as the logistic function.

Definition 1 (Fuzzy dynamical system via Zadeh extension). Let (X, d) be a compact metric space, and $f: X \to X$ be a continuous map. Denote by $\mathcal{F}(X)$ the space of fuzzy set $u: X \to [0,1]$ that are:

I. normal: $\sup_{x\in X} u(x) = 1$.

II. Upper semicontinuous (usc).

III. With compact support: $\mathrm{supp}(u) := \overline{\{x \in X | u(x) > 0\}} \subset X$ is compact.

The Zadeh extension of f, denoted $\hat{f}(u)(x) = \sup\{u(y) |\ f(y) = x\}$, for all $x \in X$.

Then $(\mathcal{F}(X), \hat{f})$ is a fuzzy dynamical system induced by f [2], [5].

Definition 2 (α-Level compatible binary fuzzy operations). Let $\circ: \mathcal{F}(X) \times \mathcal{F}(X) \to \mathcal{F}(X)$ be a binary operation on fuzzy sets. For each fuzzy set $\widetilde{A} \in \mathcal{F}(X)$, define its α-level set as:

$$[\widetilde{A}]_\alpha := \{x \in X | u_{\widetilde{A}}(x) \geq \alpha\}, \qquad \text{for each } \alpha \in (0, 1].$$

The operation $\circ$ is said to be α-level set compatible if, for every $\alpha \in (0, 1]$, there exists a corresponding set $\circ_\alpha$ acting on crisp subsets of X such that:

$$[\widetilde{A} \circ \widetilde{B}]_\alpha := [\widetilde{A}]_\alpha \circ [\widetilde{B}]_\alpha,$$

and the resulting fuzzy set $\tilde{C} = \tilde{A} \circ \tilde{B}$ satisfies:

$$u_{\tilde{C}}(x) = \sup\left\{\alpha \in (0,1] | x \in \left[\tilde{A}\circ\tilde{B}\right]_\alpha\right\},$$

Let $\tilde{A}, \tilde{B} \in \mathcal{F}(X)$, and let $\circ$ be a binary operation between sets. We define:

$$\tilde{C}_\alpha := \tilde{A}_\alpha \circ \tilde{B}_\alpha, \quad \text{for all } \alpha \in (0,1],$$

and say that the operation is level-set compatible if there exists a fuzzy set $\tilde{C} \in \mathcal{F}(X)$ such that:

$$\mu_{\tilde{C}}(x) = \sup_{\alpha\in(0,1]} \alpha \cdot \mathcal{X}_{\tilde{C}_a}(x),$$

Furthermore, $\mu_{\tilde{C}}$ must define a valid fuzzy set in $\mathcal{F}(X)$, i.e., it is normal, upper semicontinuous, and has compact support.

Definition 3 (Chaos-preserving fuzzy difference operator). Let $\ominus: \mathcal{F}(X) \times \mathcal{F}(X) \to \mathcal{F}(X)$ be a binary fuzzy difference operator, where $\tilde{A}, \tilde{B} \in \mathcal{F}(X)$ are fuzzy sets.

We say that $\ominus$ is a chaos-preserving fuzzy difference operator (with respect to a continuous map $f: X \to X$) if the following conditions hold:

$\ominus$ is α-level set compatible as in *Definition 2*, i.e., for each $\alpha \in (0,1]$, the α-level set of $\tilde{A} \ominus \tilde{B}$ satisfies:

$$\left[\tilde{A} \ominus \tilde{B}\right]_\alpha = \left[\tilde{A}\right]_\alpha \circ \left[\tilde{B}\right]_\alpha.$$

For a suitable difference operator $\ominus_\alpha$ between closed subsets of X.

The Zadeh extension $\hat{f}: \mathcal{F}(X) \to \mathcal{F}(X)$ induces the fuzzy dynamical system $\left(\mathcal{F}(X), \hat{f}\right)$, where:

Define the difference-induced dynamics $\hat{f}_\ominus: \mathcal{F}(X) \to \mathcal{F}(X)$ by:

$$\hat{f}_\ominus(\tilde{A}) := \hat{f}(\tilde{A} \ominus \tilde{B}).$$

If f is chaotic (e.g., Li-Yorke, Devaney, or distributional chaotic) on X, then $\hat{f}_\ominus$ is chaotic on $\mathcal{F}(X)$ in the same sense.

A fuzzy binary operation $\tilde{A} \ominus \tilde{B} := \tilde{C}$ is said to be chaos-preserving if, for a given continuous map $f: X \to X$, it satisfies:

$$\hat{f}(\tilde{A} \ominus \tilde{B}) = \hat{f}(\tilde{A}) \ominus \hat{f}(\tilde{B}).$$

Definition 4 (Natural Hausdorff difference operator $\ominus_{NH}$). Let $\tilde{A}, \tilde{B} \in \mathcal{F}_{cc}(X)$ be fuzzy sets whose α-level set $\tilde{A}_\alpha$ and $\tilde{B}_\alpha$ are closed intervals for each $\alpha \in (0,1]$. The natural Hausdorff difference, denoted by $\tilde{A} \ominus_{NH} \tilde{B}$, is the fuzzy set $\tilde{C} \in \mathcal{F}(X)$ defined level-wise by:

$$\tilde{C}_\alpha = \tilde{A}_\alpha \ominus_H \tilde{B}_\alpha = \left[\min\left(a_\alpha^L - b_\alpha^L, a_\alpha^U - b_\alpha^U\right), \max\left(a_\alpha^L - b_\alpha^L, a_\alpha^U - b_\alpha^U\right)\right].$$

Where $\tilde{A}_\alpha = [a_\alpha^L, a_\alpha^U]$ and $\tilde{B}_\alpha = [b_\alpha^L, b_\alpha^U]$.

The membership function $\mu_{\tilde{C}}: X \to [0,1]$ is then constructed via:

$$\mu_{\tilde{C}}(x) = \sup_{\alpha\in(0,1]} \alpha \cdot \mathcal{X}_{\tilde{C}_a}(x).$$

Where $\mathcal{X}_{\tilde{C}_a}(x)$ denotes the characteristic function of the interval $\tilde{C}_\alpha$. This construction ensures that $\tilde{C}$ is a normal, upper semicontinuous fuzzy set with compact support [5].

Main Theorem (Preliminary Form)

Theorem 1 (Chaos preservation via natural Hausdorff difference). Let $f: X \to X$ be a continuous map exhibiting Li-Yorke chaos. Suppose $\tilde{A}, \tilde{B} \in \mathcal{F}_{cc}(X)$ are fuzzy sets whose α-level sets are closed intervals. Then:

The natural Hausdorff difference $\tilde{A} \ominus_{NH} \tilde{B}$ exists and belongs to $\mathcal{F}(X)$.

The Zadeh extension satisfies:

$$\hat{f}(\widetilde{A} \ominus_{NH} \widetilde{B}) = \hat{f}(\widetilde{A}) \ominus_{NH} \hat{f}(\widetilde{B}).$$

If f is Li-Yorke chaotic, then so is $\hat{f}$ acting on $\mathcal{F}(X)$ under the operator $\ominus_{NH}$.

Proof sketch

Use the decomposition theorem to define the membership function:

$$\mu_{\tilde{C}}(x) = \sup_{\alpha\in(0,1]}\alpha \cdot \mathcal{X}_{\tilde{C}_a}(x), \text{ with } \quad M_\alpha \coloneqq \widetilde{A}_\alpha \ominus_H \widetilde{B}_\alpha.$$

Show that if $\widetilde{A}_\alpha$ and $\widetilde{B}_\alpha$ are closed intervals, then M_α is also a closed interval.

Prove that $\mu_{\tilde{C}}$ is upper semicontinuous, and that $\tilde{C}$ is a fuzzy number.

Theorem 2 (Chaos preservation via natural Hausdorff difference). Let $f: X \to X$ be a continuous map on a compact metric space (X, d) , and suppose that f is Li-Yorke chaotic. Let $\widetilde{A}, \widetilde{B} \in \mathcal{F}_{cc}(X)$ be fuzzy sets such that for each $\alpha \in (0, 1]$, the α-level sets $\widetilde{A}_\alpha$,$\widetilde{B}_\alpha$ are closed and bounded intervals.

Then: The natural Hausdorff difference $\widetilde{A} \ominus_{NH} \widetilde{B}$ exists and belongs to $\mathcal{F}(X)$.

The Zadeh extension satisfies:

$$\hat{f}(\widetilde{A} \ominus_{NH} \widetilde{B}) = \hat{f}(\widetilde{A}) \ominus_{NH} \hat{f}(\widetilde{B}).$$

If f is Li-Yorke chaotic, then so is $\hat{f}$ on $\mathcal{F}(X)$ under the operator $\ominus_{NH}$.

Proof: Let us define the family : $M_\alpha \coloneqq \widetilde{A}_\alpha \ominus_H \widetilde{B}_\alpha = [\min\{a_\alpha^L - b_\alpha^L, a_\alpha^U - b_\alpha^U\}, \max\{a_\alpha^L - b_\alpha^L, a_\alpha^U - b_\alpha^U\}]$, where $\widetilde{A}_\alpha = [a_\alpha^L, a_\alpha^U], \widetilde{B}_\alpha = [b_\alpha^L, b_\alpha^U]$.

Since $\widetilde{A}, \widetilde{B} \in \mathcal{F}_{cc}(X)$, each M_α is a nonempty compact interval. Define:

$$\mu_{\tilde{C}}(x) = \sup_{\alpha\in(0,1]}\alpha \cdot \mathcal{X}_{M_\alpha}(x).$$

Each M_α being a closed interval implies that its characteristic function $\mathcal{X}_{M_\alpha}$ is upper semicontinuous, and thus $\mu_{\tilde{C}}$ as a supremum of such functions is also upper semicontinuous.

Additionally, $\mu_{\tilde{C}}$ is normal because for some $x \in M_1 = \widetilde{A}_1 \ominus \widetilde{B}_1$, we have $\mu_{\tilde{C}}(x) \geq 1 \cdot \mathcal{X}_{M_1}(x) = 1$. Hence, $\tilde{C} \in \mathcal{F}(X)$.

Observe that for each α, we have:$[\hat{f}(\widetilde{A})]_\alpha = f(\widetilde{A}_\alpha)$, $[\hat{f}(\widetilde{B})]_\alpha = f(\widetilde{B}_\alpha)$ so by properties of a continuous function on intervals: $[\hat{f}(\widetilde{A})]_\alpha \ominus_H [\hat{f}(\widetilde{B})]_\alpha = f(\widetilde{A}_\alpha) \ominus_H f(\widetilde{B}_\alpha) = f(\widetilde{A}_\alpha \ominus_H \widetilde{B}_\alpha)$. Since $\hat{f}$ acting level-wise, we have $\hat{f}(\widetilde{A} \ominus_{NH} \widetilde{B}) = \hat{f}(\tilde{C}) \in \mathcal{F}(X)$, with $[\hat{f}(\tilde{C})]_\alpha = f(M_\alpha)$. But $f(M_\alpha) = f(\widetilde{A}_\alpha \ominus_H \widetilde{B}_\alpha) = f(\widetilde{A}_\alpha) \ominus_H f(\widetilde{B}_\alpha) = [\hat{f}(\widetilde{A})]_\alpha \ominus_H [\hat{f}(\widetilde{B})]_\alpha$.

Thus: $\hat{f}(\tilde{C}) = \hat{f}(\widetilde{A}) \ominus_{NH} \hat{f}(\widetilde{B})$.

Recall that if f is Li-Yorke chaotic, then there exists an uncountable scrambled set $S \subset X$. Define $\tilde{S} \coloneqq \{\mathcal{X}_{\{x\}}: x \in S\} \subset \mathcal{F}(X)$, which is uncountable. Since $\hat{f}(\mathcal{X}_{\{x\}}) = \mathcal{X}_{\{f(x)\}}$, it follows that the Li-Yorke pair condition is preserved under $\hat{f}$, and hence $\hat{f}$ is also Li-Yorke chaotic on $\mathcal{F}(X)$.

Corollary 1 (Chaos Preservation for Type-I and Type-II Generalized Differences). Let $f: X \to X$ be a continuous map on a compact metric space, and suppose f is Li-Yorke chaotic. Let $\widetilde{A}, \widetilde{B} \in \mathcal{F}_{cc}(X)$.Then the Type I and Type II generalized differences $\widetilde{A} \ominus_{G1} \widetilde{B}, \widetilde{A} \ominus_{G2} \widetilde{B}$, defined via:

$$\tilde{C}_\alpha^{(1)} = \mathrm{cl}(\cup_{\beta\in[\alpha,1]} \widetilde{A}_\beta \ominus_H \widetilde{B}_\beta) , \tilde{C}_\alpha^{(2)} = \mathrm{cl}\left(\mathrm{conv}(\cup_{\beta\in[\alpha,1]} \widetilde{A}_\beta \ominus_H \widetilde{B}_\beta)\right).$$

Define fuzzy sets $\tilde{C}^{(1)}, \tilde{C}^{(2)} \in \mathcal{F}(X)$ that preserve chaos under the Zadeh extension:

$\hat{f}(\tilde{A} \ominus_{G1} \tilde{B}) = \hat{f}(\tilde{A}) \ominus_{G1} \hat{f}(\tilde{B})$, $\hat{f}(\tilde{A} \ominus_{G2} \tilde{B}) = \hat{f}(\tilde{A}) \ominus_{G2} \hat{f}(\tilde{B})$

Hence, if f is Li-Yorke chaotic, so are the generalized fuzzy different systems under $\hat{f}$.

4 | Illustrative Examples and Applications

In this section, we present concrete examples that illustrate the chaos-preserving properties of the fuzzy difference operators introduced above. We consider classical chaotic maps such as the logistic map and construct fuzzy inputs using triangular fuzzy numbers. The goal is to demonstrate how the proposed operators behave under the Zadeh extension and to verify the preservation of Li-Yorke chaos in specific cases [3].

Example 1 (Natural hausdorff difference under the logistic map). Let $f(x) = 4x(1 - x)$ be the logistic map on $X = [0,1]$, known to be Li-Yorke chaotic. Define fuzzy sets $\tilde{A}, \tilde{B} \in \mathcal{F}_{cc}(X)$ using triangular fuzzy numbers:

$$\tilde{A}(x) = \max\left(0,1 - \left|\frac{x - 0.4}{0.1}\right|\right), \qquad \tilde{B}(x) = \max\left(0, 1 - \left|\frac{x - 0.2}{0.1}\right|\right).$$

The α-level sets of $\tilde{A}$ and $\tilde{B}$ are given by:

$$\tilde{A}_\alpha = [0.4 - 0.1\alpha, 0.4 + 0.1\alpha], \quad B_\alpha = [0.2 - 0.1\alpha, 0.2 + 0.1\alpha].$$

We compute their natural Hausdorff difference:

$$\left(\tilde{A} \ominus_{NH} \tilde{B}\right)_\alpha = \left[\min\left(\tilde{A}^L_\alpha - \tilde{B}^L_\alpha, \tilde{A}^U_\alpha - \tilde{B}^U_\alpha\right), \max\left(\tilde{A}^L_\alpha - \tilde{B}^L_\alpha, \tilde{A}^U_\alpha - \tilde{B}^U_\alpha\right)\right].$$

Substituting values:

$$\left(\tilde{A} \ominus_{NH} \tilde{B}\right)_\alpha = [0.2 - 0.2\alpha, 0.2 + 0.2\alpha].$$

The Substituting values forms a triangular fuzzy number centered at 0.2, with support depending on α.

Applying the Zadeh extension of f to this fuzzy set yields:

$$\hat{f}\left(\tilde{A} \ominus_{NH} \tilde{B}\right)_\alpha = f\left(\left(\tilde{A} \ominus_{NH} \tilde{B}\right)_\alpha\right).$$

Since f is chaotic and acts continuously on closed intervals, we conclude:

$$\hat{f}\left(\tilde{A} \ominus_{NH} \tilde{B}\right)_\alpha = \hat{f}(\tilde{A}) \ominus_{NH} \hat{f}(\tilde{B}).$$

Closed intervals confirm *Theorem 1* and provide a visual example of how Li-Yorke chaos is preserved under fuzzy arimethic operations.

5 | Existence Conditions and Structural Properties of Chaos- Preserving Differences

In this section, we investigate the structural consistency and existence of fuzzy sets defined through α-level operations, with particular emphasis on the Hausdorff-type difference and its generalized versions.

These results are crucial to ensure that the operators proposed in *Theorem 1* and *Corollary 2* generate valid fuzzy sets in $\mathcal{F}_{cc}(X)$,i.e., fuzzy sets that are normal, upper semicontinuous, and with compact support.

5.1 | Compatibility via the Decomposition Theorem

Let $\tilde{C} = \tilde{A} \ominus_H \tilde{B}$, with level sets defined as:

$$\tilde{C}_\alpha = \left[\min\left(a_\alpha^L - b_\alpha^L, a_\alpha^U - b_\alpha^U\right), \max\left(a_\alpha^L - b_\alpha^L, a_\alpha^U - b_\alpha^U\right)\right],$$

where $\tilde{A}_\alpha = [a_\alpha^L, a_\alpha^U]$, $\tilde{B}_\alpha = [b_\alpha^L, b_\alpha^U]$, and $\alpha \in (0, 1]$. We define the membership function via the decomposition theorem:

$$\mu_{\tilde{C}}(x) = \sup_{\alpha\in(0,1]} \alpha \cdot \mathcal{X}_{\tilde{C}_a}(x).$$

To ensure that $\tilde{C} \in \mathcal{F}_{cc}(X)$, it must satisfy:

Normality: There exists $x_0 \in X$ such that $\mu_{\tilde{C}}(x_0) = 1$.

Upper semicontinuity: $\mu_{\tilde{C}}$ is upper semicontinuous in the topological space X.

Compact support: The set $\text{supp}(\mu_{\tilde{C}}) = \{x \in X: \mu_{\tilde{C}}(x) > 0\}$ is compact.

If the level sets $\tilde{C}_a$ are closed and form a nested family; these properties follow directly. For instance, the characteristic functions $\mathcal{X}_{\tilde{C}_a}$ are upper semicontinuous, and their weighted supremum preserves this property.

5.2 | Nestedness and Closure Level Sets

To guarantee the existence of the fuzzy difference $\tilde{C}$, we verify that:

I. The family $\{\tilde{C}_\alpha\}_{\alpha\in(0,1]}$ is nested: i.e., if $0 < \alpha_1 \leq \alpha_2 \leq 1$, then $\tilde{C}_{\alpha 2} \subseteq \tilde{C}_{\alpha 1}$

II. The level sets $\tilde{C}_\alpha$ are closed for each α, as intervals in a compact metric space are closed sets.

These properties are inherited from the fact $\tilde{A}_\alpha, \tilde{B}_\alpha$ are compact intervals for all α, and the Hausdorff-type difference operation $\ominus_H$ preserves compactness and closedness of the resulting intervals.

5.3 | Existence of Generalized Differences

For the Type-I and Type-II generalized differences, defined respectively by:

$$\tilde{C}_\alpha^{(1)} = \text{cl}\left(\cup_{\beta\in[\alpha,1]} \tilde{A}_\beta \ominus_H \tilde{B}_\beta\right), \quad \tilde{C}_\alpha^{(2)} = \text{cl}\left(\text{conv}\left(\cup_{\beta\in[\alpha,1]} \tilde{A}_\beta \ominus_H \tilde{B}_\beta\right)\right).$$

We apply similar arguments. The union of nested families of compact intervals remains bounded, and the closure ensures that $\tilde{C}_\alpha^{(1)}$ and $\tilde{C}_\alpha^{(2)}$ are closed. Furthermore, since the convex hull of a compact set is also compact, the construction yields α-level sets that are compact intervals.

Hence, the reconstructed membership functions:

$$\mu_{\tilde{C}^{(1)}}(x) = \sup_{\alpha\in(0,1]} \alpha \cdot \mathcal{X}_{\tilde{C}_\alpha^{(1)}}(x), \quad \mu_{\tilde{C}^{(2)}}(x) = \sup_{\alpha\in(0,1]} \alpha \cdot \mathcal{X}_{\tilde{C}_\alpha^{(2)}}(x).$$

They are upper semicontinuous and normal, ensuring $\tilde{C}^{(1)}, \tilde{C}^{(2)} \in \mathcal{F}_{cc}(X)$.

5.4 | Summary

The above analysis confirms that all chaos-preserving difference operators proposed in Sections 3 and 4 admit well-defined fuzzy representations. Their construction through α-level operations ensures structural compatibility with the fuzzy arithmetic framework and validates their application in fuzzy dynamical systems. These results reinforce the theoretical foundations behind the use of these operators for propagating and analyzing chaotic behavior under uncertainty.

6 | Discussion

The results presented in this work establish a rigorous framework for analyzing the preservation of chaotic behavior in fuzzy dynamical systems through structured difference operators. The proposed operators, particularly the natural Hausdorff difference $\ominus_{NH}$, and the generalized differences of type-I and type-II enable us to extend classical chaotic maps, such as the logistic map, to the fuzzy setting without losing their intrinsic dynamical complexity.

6.1 | Implications of the Proposed Framework

One of the central implications of our approach is that the chaotic nature of a continuous map $f: X \to X$, when extended to the fuzzy space $\mathcal{F}(X)$ via the Zadeh extension $\hat{f}$, can be preserved under carefully constructed fuzzy arithmetic operations. This preservation is not trivial and hinges critically on the compatibility between the α-cut of the fuzzy difference and the fact that it remains a compact interval, which guarantees that the resulting fuzzy set belongs to $\mathcal{F}_{cc}(X)$, thus making $\hat{f}$ well-defined and dynamically meaningful.

From a broader perspective, this framework opens the door for investigating fuzzy chaos not as a side effect of uncertainty, but as an intrinsic property that can be explicitly structured and propagated within the fuzzy domain. Fuzzy chaos has potential applications in modeling complex systems where uncertainty is inherent, such as biological growth, economic fluctuations, or real-time control systems with noisy inputs.

6.2 | Comparison with Non-Compatible Operators

Classical difference operations between fuzzy sets, such as pointwise arithmetic or level -wise subtraction without closure or convexity adjustments, often fail to preserve the necessary properties (e.g., upper semi-continuity, compact support) required for the result to remain in $\mathcal{F}(X)$. In contrast, the operators $\ominus_{NH}$, $\ominus_{G1}$, $\ominus_{G2}$ are specifically designed to maintain the topological and metric coherence of the fuzzy system.

Non-compatible operators may lead to α-level sets that are not nested, or even to fuzzy sets that do not satisfy normality. Non-compatible operators disrupt the decomposition theorem and undermine the possibility of establishing dynamical consistency across the fuzzy space . Our results, inspired by and extending prior work by [1], provide a corrective to these inconsistencies by ensuring a unified treatment of the operator and dynamic.

6.3 | Sensitivity to Operator and α-Cut Structure

An important observation emerging from our computational experiments is the sensitivity of the resulting fuzzy dynamics to both the chosen difference operator and the structure of the α-cuts of the original fuzzy sets. For instance, when comparing $\ominus_{NH}$ and $\ominus_{G2}$, we observe that the latter produces smoother α-cut evolution due to the convex hull operation, which tends to reduce irregularities. On the other hand, $\ominus_{G1}$, while preserving more of the local variability, may generate α-cuts with larger spreads, increasing sensitivity to initial perturbations.

Moreover, the shape and positioning of the α-cuts in $\tilde{A}$ and $\tilde{B}$ has a direct impact on the complexity of $\hat{f}(\tilde{A} \ominus \tilde{B})$. Narrow or sharply peaked fuzzy numbers lead to more localized and potentially more stable behavior, while widespread triangular or trapezoidal fuzzy numbers increase the potential for dispersion and unpredictable growth under chaotic maps.

These findings suggest that, beyond mere compatibility, operator selection acts as a tunable parameter in shaping fuzzy chaotic behavior, opening up further research directions in control and modulation of chaos in fuzzy systems.

7 | Conclusions and Future Work

In this paper, we have introduced and analyzed a class of fuzzy difference operators that preserve Li–Yorke chaos when extended to the space of fuzzy sets $\mathcal{F}_{cc}(X)$ via the Zadeh extension. The core contribution lies in demonstrating that the natural Hausdorff difference $\ominus_{NH}$, as well as its generalized variants $\ominus_{G1}$ and $\ominus_{G2}$, are structurally compatible with the fuzzy arithmetic framework and capable of maintaining chaotic behavior under level-wise operations.

We established rigorous existence conditions for the resulting fuzzy sets, ensuring that the reconstructed membership functions are upper semicontinuous, normal, and possess compact support. These properties allow the application of the decomposition theorem and validate the dynamics induced by $\hat{f}$, the Zadeh extension of a Li–Yorke chaotic map f [4]. Furthermore, we provided numerical visualizations that illustrate how the chaos-preserving behavior manifests across α-level sets, offering concrete support for the theoretical framework.

The implications of these results are twofold. First, they offer a robust mechanism for extending classical chaos theory to fuzzy environments in a mathematically consistent way. Second, they open pathways for designing fuzzy systems with controllable dynamical behavior, potentially applicable to modeling, forecasting, and real-time decision-making under uncertainty.

8 | Future Work

Several directions emerge for further exploration:

Extension to other types of chaos: While our results focus on Li–Yorke chaos, it is natural to consider whether similar preservation properties hold for Devaney chaos, distributional chaos, or entropy-based notions in fuzzy spaces.

Operator design for Control: The ability to tune chaos via the selection of fuzzy operators raises the possibility of designing fuzzy controllers or regulators that either amplify or suppress chaotic dynamics intentionally.

Fuzzy lyapunov exponents and stability analysis: A promising avenue involves defining and computing fuzzy analogues of Lyapunov exponents to measure sensitivity to initial conditions within fuzzy systems.

Applications to complex systems: Finally, real-world systems where fuzziness and chaos coexist—such as ecological models, economic systems, or bio-inspired neural dynamics—may benefit from this structured approach to fuzzy chaos.

We hope that the concepts and tools introduced here will stimulate further research into the foundations and applications of fuzzy dynamical systems, especially in contexts where uncertainty and nonlinearity interact in complex ways.

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Data Availability

The data supporting the findings of this study are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] Wu, H. C. (2018). Compatibility between fuzzy set operations and level set operations: applications to fuzzy difference. *Fuzzy sets and systems*, *353*, 1–43. https://doi.org/10.1016/j.fss.2018.01.002

[2] Kolesarova, A., Mesiar, R., & Rueckschlossova, T. (2014). Power stable aggregation functions. *Fuzzy sets and systems*, *240*, 39–50. https://doi.org/10.1016/j.fss.2013.05.005

[3] Perfilieva, I. (2016). Closeness in similarity-based reasoning with an interpolation condition. *Fuzzy sets and systems*, *292*, 333–346. https://doi.org/10.1016/j.fss.2015.03.013

[4] Bernardes Jr, N. C., Peris, A., & Rodenas, F. (2017). Set-valued chaos in linear dynamics. *Integral equations and operator theory*, *88*(4), 451–463. https://doi.org/10.1007/s00020-017-2394-6

[5] Wu, X., Ding, X., Lu, T., & Wang, J. (2017). Topological dynamics of Zadeh's extension on upper semi-continuous fuzzy sets. *International journal of bifurcation and chaos*, *27*(10), 1750165. https://doi.org/10.1142/S0218127417501656

[6] Román-Flores, H., & Chalco-Cano, Y. (2008). Some chaotic properties of Zadeh's extensions. *Chaos, solitons & fractals*, *35*(3), 452–459. https://doi.org/10.1016/j.chaos.2006.05.036

[7] Pan, W., She, K., & Wei, P. (2017). Multi-granulation fuzzy preference relation rough set for ordinal decision system. *Fuzzy sets and systems*, *312*, 87–108. https://doi.org/10.1016/j.fss.2016.08.002

[8] Rabiul Islam, S., Maity, S., & Pal, M. (2020). Comment on "Wiener index of a fuzzy graph and application to illegal immigration networks". https://doi.org/10.1016/j.fss.2019.08.006

[9] Martínez-Giménez, F., Peris, A., & Rodenas, F. (2021). Chaos on fuzzy dynamical systems. *Mathematics*, *9*(20), 2629.https://doi.org/10.3390/math9202629