Optimality

www.opt.reapress.com

Opt. Vol. 2, No. 1 (2025) 43–51.

Paper Type: Original Article

The Computational Subgroups for the Finite Fuzzy Nilpotent Groups Involving Indetermi-Nates (Varying) $m; n$

Sunday Adesina Adebisi[1], Mike Ogiugo[2,*], Michael Enioluwafe[3]

[1] Department of Mathematics, Faculty of Science, University of Lagos, Nigeria; adesinasunday71@gmail.com.
[2] Department of Mathematics, School of Science, Yaba College of Technology, Nigeria; ekpenogiugo@gmail.com.
[3] Department of Mathematics, Faculty of Science, University of Ibadan, Nigeria; michael.enioluwafe@gmail.com.

Citation:

Received: 10 October 2024 *Revised: 18 December 2024* *Accepted: 28 February 2025*	Adebisi, S. A., Ogiugo, M., & Enioluwafe, M. (2025). The computational subgroups for the finite fuzzy nilpotent groups involving indetermi-nates (varying) m; n. *Optimality*, 2(1), 43-51.

Abstract

This theory of fuzzy sets has a wide range of applications, one of which is that of fuzzy groups . The Fuzzy sets were actually been introduced by Zadeh. Even though, the story of Fuzzy logic started much earlier, it was specially designed mathematically to represent uncertainty and vagueness. It was also, to provide formalized tools for dealing with the imprecision intrinsic to many problems. The term fuzzy logic is generic as it can be used to describe the likes of fuzzy arithmetic, fuzzy mathematical programming, fuzzy topology, fuzzy graph theory and fuzzy data analysis which are customarily called fuzzy set theory. A group is nilpotent if it has a normal series of a finite length n. By this notion, every finite p-group is nilpotent. The nilpotence property is an hereditary one. Thus, every finite p-group possesses certain remarkable characteristics. In this paper, the explicit formulae is given for the number of distinct fuzzy subgroups of the Cartesian product of the dihedral group of order eight with a cyclic group of order of an m power of two for, which m is not less than three .

Keywords: Finite p-Groups, Nilpotent Group, Fuzzy subgroups, Dihedral Group, Inclusion-Exclusion Principle,Maximal subgroups.

1|Introduction

The aspect of pure Mathematics has undergone a lot of dynamic developments over the years . Concerning the theory of fuzzy group , the classification, most especially the finite p-groups cannot be overlooked. For instance, many researchers have treated cases of finite abelian groups. Since inception , the study has been extended

Corresponding Author: ekpenogiugo@gmail.com

to some other important classes of finite abelian and nonabelian groups such as the dihedral , quaternion, semidihedral, and hamiltonian groups. Other different approaches have been so far, applied for the classification. The Fuzzy sets were introduced by Zadeh in 1965. Even though, the story of Fuzzy logic started much more earlier, it was specially designed mathematically to represent uncertainty and vagueness. It was also, to provide formalized tools for dealing with the imprecision intrinsic to many problems. The term fuzzy logic is generic as it can be used to describe the likes of fuzzy arithmetic, fuzzy mathematical programming, fuzzy topology, fuzzy graph theory ad fuzzy data analysis which are customarily called fuzzy set theory. This theory of fuzzy sets has a wide range of applications, one of which is that of fuzzy groups developed by Rosenfield in 1971. This by far, plays a pioneering role for the study of fuzzy algebraic structures. Other notions have been developed based on this theory. These, amongst others, include the notion of level subgroups by P.S. Das used to characterize fuzzy subgroups of finite groups and that of equivalence of fuzzy subgroups introduced by Murali and Makamba which we use in this work. (Please, see [1 - 9])
By the way, A group is nilpotent if it has a normal series of a finite length n.

$$G = G_0 \geq G_1 \geq G_2 \geq \cdots \geq G_n = \{e\},$$

where

$$G_i/G_{i+1} \leq Z(G/G_{i+1}).$$

By this notion, every finite p-group is nilpotent. The nilpotence property is an hereditary one. Thus,

(i) Any finite product of nilpotent group is nilpotent.

(ii) If G is nilpotent of a class c, then, every subgroup and quotient group of G is nilpotent and of class $\leq c$.

The problem of classifying the fuzzy subgroups of a finite group has so far experienced a very rapid progress. One particular case or the other have been treated by several papers such as the finite abelian as well as the non-abelian groups. The number of distinct fuzzy subgroups of a finite cyclic group of square-free order has been determined. Moreover, a recurrence relation is indicated which can successfully be used to count the number of distinct fuzzy subgroups for two classes of finite abelian groups. They are the arbitrary finite cyclic groups and finite elementary abelian p-groups. For the first class, the explicit formula obtained gave rise to an expression of a well-known central Delannoy numbers. Some forms of propositions for classifying fuzzy subgroups for a class of finite p-groups have been made by Marius Tarnauceaus. It was from there, the study was extended to some important classes of finite non-abelian groups such as the dihedral and hamiltonian groups. And thus, a method of determining the number and nature of fuzzy subgroups was developed with respect to the equivalence relation. There are other different approaches for the classification. The corresponding equivalence classes of fuzzy subgroups are closely connected to the chains of subgroups, and an essential role in solving counting problem is again played by the inclusion - exclusion principle. This hereby leads to some recurrence relations, whose solutions have been easily found. For the purpose of using the Inclusion - Exclusion principle for generating the number of fuzzy subgroups, the finite p-groups has to be explored up to the maximal subgroups. The responsibility of describing the fuzzy subgroup structure of the finite nilpotent groups is the desired objective of this work. Suppose that $(G, \cdot, e)$ is a group with identity e. Let $S(G)$ denote the collection of all fuzzy subsets of G. An element $\lambda \in S(G)$ is called a fuzzy subgroup of G whenever it satisfies some certain given conditions . Such conditions are as follows :

(i) $\lambda(ab) \geq\in \{\lambda(a), \lambda(b)\}, \ \forall\, a, b \in G$;

(ii) $\lambda(a^{-1} \geq \lambda(a)$ for any $a \in G$.

And, since $(a^{-1})^{-1} = a$, we have that $\lambda(a^{-1}) = \lambda(a)$, for any $a \in G$.
Also, by this notation and definition, $\lambda(e) = \sup \lambda(G)$. [Marius [6]].
Theorem : The set $FL(G)$ possessing all fuzzy subgroups of G forms a lattice under the usual ordering of fuzzy set inclusion. This is called the fuzzy subgroup lattice of G.

We define the level subset:

$$\lambda G_\beta = \{a \in G/\lambda(a) \geq \beta\} \text{ for each } \beta \in [0, 1]$$

The fuzzy subgroups of a finite p-group G are thus, characterized, based on these subsets. In the sequel, λ is a fuzzy subgroup of G if and only if its level subsets are subgroups in G. This theorem gives a link between $FL(G)$ and $L(G)$, the classical subgroup lattice of G.F

Moreover, some natural relations on $S(G)$ can also be used in the process of classifying the fuzzy subgroups of a finite q-group G. One of them is defined by: $\lambda \sim \gamma$ iff $(\lambda(a) > \lambda(b) \iff v(a) > v(b), \ \ \forall\ a, b \in G)$. Alos, two fuzzy subgroups λ, γ of G and said to be distinct if $\lambda \times v$.

As a result of this development, let G be a finite p-group and suppose that $\lambda : G \longrightarrow [0,1]$ is a fuzzy subgroup of G. Put $\lambda(G) = \{\beta_1, \beta_2, \ldots, \beta_k\}$ with the assumption that $\beta_1 < \beta_2 > \cdots > \beta_k$. Then, ends in G is determined by λ.

$$\lambda G_{\beta_1} \subset \lambda G_{\beta_2} \subset \cdots \subset \lambda G_{\beta_k} = G \qquad (a)$$

Also, we have that:

$$\lambda(a) = \beta_t \iff t = \max\{r/a \in \lambda G_{\beta_r}\} \iff a \in \lambda G_{\beta_t} \backslash \lambda G_{\beta_{t-1}},$$

for any $a \in G$ and $t = 1, \ldots, k$, where by convention, set $\lambda G_{\beta_0} = \phi$.

2|Methodology

We are going to adopt a method that will be used in counting the chains of fuzzy subgroups of an arbitrary finite p-group G is described. Suppose that $M_1, M_2, \ldots, M_t$ are the maximal subgroups of G, and denote by $h(G)$ the number of chains of subgroups of G which ends in G. By simply applying the technique of computing $h(G)$, using the application of the Inclusion-Exclusion Principle, we have that:

$$h(G) = 2\left(\sum_{r=1}^{t} h(M_r) - \sum_{1 \le r_1 < r_2 \le t} h(M_{r_1} \cap M_{r_2}) + \cdots + (-1)^{t-1} h\left(\bigcap_{r=1}^{t} M_r\right)\right) (\#)$$

In [6], (#) was used to obtain the explicit formulas for some positive integers n.

Theorem [1] [Marius]: The number of distinct fuzzy subgroups of a finite p-group of order p^n which have a cyclic maximal subgroup is:

(i) $h(\mathbb{Z}_{p^n}) = 2^n$, (ii) $h(\mathbb{Z}_p \times \mathbb{Z}_{p^{n-1}}) = 2^{n-1}[2 + (n-1)p]$

3 The distinct Number of The Fuzzy Subgroups of The Nilpotent Group of $(D_{2^3} \times C_{2^m})$ For $m \ge 3$

Proposition 1 (see [13]) : Suppose that $G = \mathbb{Z}_4 \times \mathbb{Z}_{2^n}, n \ge 2$. **Then,** $h(G) = 2^n[n^2 + 5n - 2]$

Proof : G has three maximal subgroups of which two are isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{2^n}$ and the third is isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}$.

Hence, $h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) = 2h(\mathbb{Z}_2 \times \mathbb{Z}_{2^n}) + 2^1 h(\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-1}}) + 2^2 h(\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}})$
$+ 2^3 h(\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-3}}) + 2^4 h(\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-4}}) + \cdots + 2^{n-2} h(\mathbb{Z}_2 \times \mathbb{Z}_{2^2})$

$$= \quad 2^{n+1}[2(n+1) + \sum_{j=1}^{n-2}[(n+1) - j]$$

$= 2^{n+1}[2(n+1) + \frac{1}{2}(n-2)(n+3)] = 2^n[n^2 + 5n - 2], n \ge 2$

We have that : $h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}) = 2^{n-1}[(n-1)^2 + 5(n-1) - 2]$
$= 2^{n-1}[n^2 + 3n - 6], n > 2$ □

Corrolary 1 : Following the last proposition, $h(\mathbb{Z}_4 \times \mathbb{Z}_{2^5}), h(\mathbb{Z}_4 \times \mathbb{Z}_{2^6}), h(\mathbb{Z}_4 \times \mathbb{Z}_{2^7})$ **and** $h(\mathbb{Z}_4 \times \mathbb{Z}_{2^8}) =$ **1536, 4096, 10496 and 26112 respectively.**

Theorem A (see [15]) : Let $G = D_{2^n} \times \mathbb{C}_2$, the nilpotent group formed by the cartesian product of the dihedral group of order 2^n and a cyclic group of order 2. Then, the number of distinct fuzzy subgroups of G is given by : $h(G) = 2^{2n}(2n+1) - 2^{n+1}, n > 3$

Proof:
The group $D_{2^n} \times C_2$, has one maximal subgroup which is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}}$, two maximal subgroups which are isomorphic to $D_{2^{n-1}} \times C_2$, and 2^2 which are isomorphic to D_{2^n}.
It thus, follows from the Inclusion-Exclusion Principle using equation,

$$\frac{1}{2}h(D_{2^n} \times C_2) = h(\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-1}}) + 4h(D_{2^n}) - 8h(D_{2^{n-1}}) - 2h(\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}}) + 2h(D_{2^{n-1}} \times C_2)$$

By recurrence relation principle we have :

$$h(D_{2^n} \times C_2) = 2^{2n}(2n+1) - 2^{n+1}, \quad n > 3$$

By the fundermental principle of mathematical induction,
set F(n) = $h(D_{2^n} \times C_2)$**, assuming the truth of F(k) =**$h(D_{2^k} \times C_2) = 2h(\mathbb{Z}_2 \times Z_{k-1})$
$+ 8h(D_{2^k} - 16hD_{2^{k-1}} - 4h(\mathbb{Z}_2 \times \mathbb{Z}_{k-2}) + 4h(D_{2^{k-1}} \times C_2) = 2^{2k}(2k+1) - 2^{k+1}$,
F(k+1) = $h(D_{2^{k+1}} \times C_2) = 2h(\mathbb{Z}_2 \times \mathbb{Z}_{2^k}) + 8h(D_{2^{k+1}} - 16h(D_{2^k} - 4h(\mathbb{Z}_2 \times \mathbb{Z}_{k-1})$
$+ 4h(D_{2^k} \times C_2) = 2^2[2^{2k}(2k-3) - 2^k]$, **which is true.** □

Proposition 2 (see [12]) : Suppose that $G = D_{2^n} \times \mathbb{C}_4$. Then, the number of distinct fuzzy subgroups of G is given by :

$$2^{2(n-2)}(64n + 173) + 3\sum_{j=1}^{n-3} 2^{(n-1+j)}(2n + 1 - 2j)$$

Proof :

$\frac{1}{2}h(D_{2^n} \times C_4) = h(D_{2^n} \times C_2) + 2h(D_{2^{n-1}} \times C_4) - 4h(D_{2^{n-1}} \times C_2) + h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}})$
$- 2h(\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-1}}) - 2h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-2}}) + 8h(\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}}) + h(\mathbb{Z}_{2^{n-1}}) - 4h(\mathbb{Z}_{2^{n-2}})$

$h(D_{2^n} \times C_4) = (n-3).2^{2n+2} + 2^{2(n-3)}(1460) + 3[2^n(2n-1) + 2^{n+1}(2n-3) + 2^{n+2}(2n-5) + \cdots + 7(2^{2(n-2)})]$

$$= (n-3).2^{2n+2} + 2^{2(n-3)}(1460) + 3\sum_{j=1}^{n-3} 2^{n-1+j}(2n+1-2j)$$
$$= 2^{2(n-2)}(64n+173) + 3\sum_{j=1}^{n-3} 2^{n-1+j}(2n+1-2j)$$

Proposition 3 (see [10]) : Let G be an abelian p-group of type $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_{p^n}$, where p is a prime and $n \geq 1$. The number of distinct fuzzy subgroups of G is
$h(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_{p^n}) = 2^n p(p+1)(n-1)(3+np+2p) + (2^n-2)p^3 - 2^{n+1}(n-1)p^3 + 2^n[p^3 + 4(1+p+p^2)]$.

Proof: There exist exactly $1 + p + p^2$ maximal subgroups for the abelian type $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_{p^n}$, [Berkovich(2008)]. One of them is isomorphic to
$\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_{p^{n-1}}$, **while each of the remaining $p + p^2$ is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$. Thus, by**

the application of the Inclusion-Exclusion Principle,we have as follows: $h(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_{p^n}) = 2^n p(p+1)(n-1)(3+np+2p) + (2^n-2)p^3 - 2^{n+1}(n-1)p^3 + 2^n[p^3 + 4(1+p+p^2)]$ **And thus,**

$$h(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_{p^{n-2}}) = 2^{n-2}[4 + (3n-5)p + (n^2-5)p^2 + (n^2-5n+8)p^3] - 2p^2.$$

□

Corrolary 2 : From (3) above, obsreve that, we are going to have that:

$$h(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{3^n}) = 2^{n+1}[18n^2 + 9n + 26] - 54$$

Similarly, for $p = 5$**, using the same analogy, we have**

$$h(\mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_{5^n}) = 2[30h(\mathbb{Z}_5 \times \mathbb{Z}_{5^n}) + h(\mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_{5^{n-1}}) - p^3 h(\mathbb{Z}_{5^n}) - 30h(\mathbb{Z}_{5^{n-1}}) + 125]$$

And for $p = 7$**,**

$$h(\mathbb{Z}_7 \times \mathbb{Z}_7 \times \mathbb{Z}_{7^n}) = 2[56h(\mathbb{Z}_7 \times \mathbb{Z}_{7^n}) + h(\mathbb{Z}_7 \times \mathbb{Z}_7 \times \mathbb{Z}_{7^{n-1}}) - 343h(\mathbb{Z}_{7^n}) - 56h(\mathbb{Z}_{7^{n-1}}) + 343]$$

We have, in general, $h(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_{p^{n-2}}) = 2^{n-2}[4 + (3n-5)p + (n^2-5)p^2 + (n^2-5n+8)p^3] - 2p^2$ □

Proposition (see [14]) :
Let $G = (D_{2^3} \times C_{2^m})$ **for** $m \geq 3$ **. Then ,** $h(G) = m(89 - 23m) + (85)2^{m+3} - 124$

Proof :
There exist seven maximal subgroups , of which one is isomorphic to $D_{2^3} \times C_{2^m-1}$**, two being isomorphic to** $C_{2^m} \times C_2 \times C_2)$**, two isomorphic to** $C_{2^m} \times C_2$**, and one each isomorphic to** $C_{2^m} \times C_4$**, and** C_{2^m} **respectively.**
Hence , by the inclusion - exclusion principle, using the propositions [1], [2], [3], and Theorem [1] we have that :

$\frac{1}{2}h(G) = h(D_{2^3} \times C_{2^{m-1}}) + 2h(C_{2^m} \times C_2) \times C_2) + 2h(C_{2^m} \times C_2) + 2h(C_{2^m} \times C_4) + h(C_{2^m}) - 12h(C_{2^m} \times C_2) - 6h(C_{2^{m-1}} \times C_2) \times C_2) - 3h(C_{2^{m-1}}) \times C_4) + 28h(C_{2^{m-1}} \times C_2) + 2h(C_{2^{m-1}} \times C_2) \times C_2) + 4h(C_{2^m} \times C_2) + h(C_{2^{m-1}} \times C_4) - 35h(C_{2^{m-1}} \times C_2) - 7h(C_{2^{m-1}} \times C_2) + h(C_{2^{m-1}} \times C_2)$
$= h(D_{2^3} \times C_2^{m-1}) + 2h(C_{2^m} \times C_2) \times C_2) - 6h(C_{2^m} \times C_2) + h(C_{2^m} \times C_4) + h(C_{2^m}) - 4h(C_{2^{m-1}} \times C_2) \times C_2) - 2h(C_{2^{m-1}}) \times C_4) + 8h(C_{2^{m-1}} \times C_2)$
$= h(D_{2^3} \times C_2^{m-1}) + 2^{m+2}(6m^2 + 7m + 9) - 32 - (6)2^m(2m+2) + 8m(2^m) - 2^{m+2}6m^2 - 5m + 8 + 2^6 + 2^m(m^2 + 5m - 2) - 2^m(3m + m^2 - 6) + 2^m = h(D_{2^3} \times C_2^{m-1}) + 2^m(46m - 4) + 2^m + 32 = h(D_{2^3} \times C_2^{m-1}) + 2^m(46m - 3) + 32$
Hence , $h(G) = 2h(D_{2^3} \times C_2^{m-1}) + 2^{m+1}(46m - 3) + 64 = 2^{m+1}(46m - 3) + 64 + 2[2^m(46m - 49) + 64 + 2h(D_{2^3} \times C_2^{m-2})] = 2^{m+1}(46m - 3) + 64 + 2^{m+1}(46m - 49) + 2^7 + 2^2 h(D_{2^3} \times C_2^{m-2}) = 2^{m+1}(46mm - 3) + 2^6 + 2^{m+1}(46m - 49) + 2^7 + 2^2[2^{m-1}(46m - 95) + 64 + 2h(D_{2^3} \times C_2^{m-3})$

$h(D_{23n} \times C_{2^m}) = (46m - 3).2^{m+1} + 2^6 + (46m - 49)2^{m+1} + 2^7 + (46m - 95)2^{m+1} + 2^8 + 2^3 h(D_{2^3} \times C_{2^{m-3}})$
$= 2^{m+1}.[(46m - 3) + (46m - 49) + (46m - 95)] + 2^6 + 2^7 + 2^8 + 2^3 h(D_{2^3} \times C_2^{m-3})$

$$= \underbrace{2^6 + 2^7 + 2^8 + \cdots + 2^{5+k}}_{\textbf{series (1)}}$$
$$+ \; 2^{m+1}.[46mk + \underbrace{(-3 - 49 - 95 \cdots (-3 - 46(k-1)))}_{\textbf{series (2)}}]$$
$$+2^k h(D_{2^3} \times C_{2^m-k}), k \in \{1, 2.3. \cdots n \in \; N\}$$

For the series (1) , we have that, $U_m = 2^6.2^{m-1} = 2^{5+k}, m+5 = k+5, \Rightarrow m = k$. **We have that** $S_{m=k} = 2^6[\frac{2^k-1}{2-1}] = 2^6(2^k.1)$

And for the second series (2), we have that , $T_m = -3 + (m-1)(-46) = -3 - 46(k-1) \Rightarrow m-1 = k-1, n = k$ **Hence ,** $S_m = k = \frac{k}{2}[2(-3) + (k-1)(-46)] = \frac{k}{2}(-6 - 46k + 46) = \frac{k}{2}(40 - 46k)$, **We have that** $h(D_{23n} \times C_{2^m}) = \frac{k}{2}(40-46k) + 2^6(2^k.1) + 2^k h(D_3 \times C_{2^m-k}$. **By setting** $m = k$ **we have that** $k = m-3$.**Hence ,** $h(D_{2^3} \times C_{2^m}) = (m-3)(20-23m) + 2^6(2^{m-3}-1) + 2^m - 3h(D_3 \times C_{2^3})$

$h(G) = (m-3)(20-23m) + 2^6(2^{m-3}-1) + 2^{m-3}(5376) = (m-3)(20-23m) + 2^{m-3} - 2^6 + 2^{m+5}(21) = 20m - 23m^2 - 60 + 69m + 2^{m+3} - 2^6 + (21)2^{m+5} = (89m - 23m^2 - 60) + 2^{m+3} - 2^6 + (21)2^{m+5} = m(89-23m) - 124 + (85)2^{m+3}$ □

Theorem (see [11]) : Let $G = \mathbb{Z}_{2^n} \times \mathbb{Z}_8$**, then** $h(G) = \frac{1}{3}(2^{n+1})(n^3 + 12n^2 + 17n - 24)$
Proof : The three maximal subgroups of G **have the following properties :**
one is isomorphic to $\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-1}}$**), while two are isomorphic to** $\mathbb{Z}_4 \times \mathbb{Z}_{2^n}$**) .**
We have : $\frac{1}{2}h(G) = 2h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) + h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-1}}) - 3h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}) + h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}})$
$= 2h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) + h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-1}}) - 2h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}})$
$= h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-1}}) + 2h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) - h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}})$
Hence , $h(G) = 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) - 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}) + 2h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-1}})$
$= 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) + 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}) + 8h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-2}}) - 16h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-3}})$
$+ 32h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-3}}) - 32h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-4}}) + 16h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-4}})$
$= 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) + 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}) + 8h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-2}}) + 16h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-3}})$
$+ 32h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-4}}) - 64h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-5}}) + 32h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-5}}) + \cdots - 2^{j+1}h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-j}})$
$+ 2^j h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-j}})$ **, for** $n - j = 3$

$$= 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) + 2^{n-3}h(\mathbb{Z}_8 \times \mathbb{Z}_{2^3}) - 2^{n-1}h(\mathbb{Z}_4 \times \mathbb{Z}_{2^3}) + \sum_{k=1}^{n-3}[2^{k+1}h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-k}})$$

$= 2^{n+2}[n^2 + 5n + 3] + \sum_{k=1}^{n-3} h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-k}}) = 2^{n+2}((n^2 + 5n + 3) + \frac{1}{6}(n-3)(n^2 + 9n + 14))$
$= \frac{1}{3}(2^{n+1})(n^3 + 12n^2 + 17n - 24), n > 2.$ □

Proposition (see [16] : Suppose that $G = D_{2^n} \times \mathbb{C}_8$**. Then, the number of distinct fuzzy subgroups of** G **is given by :**

$$2^{2(n-1)}(6n + 113) + 2^n[13 - 6n - 2n^2 + 3\sum_{j=1}^{n-3} 2^{(j-1j)}(2n + 1 - 2j)]$$

$$+\frac{1}{3}(2^{n+2})[(n-1)^3 + (n-2)^3 + 24n^2 - 38n - 30 + \sum_{k=1}^{n-5} 2^k[(n-2-k)^3 + 12(n-2-k)^2 + 17(n-k) - 58]]$$

Proof : $h(D_{2^n} \times C_8) = 2h(\mathbb{Z}_{2^{n-1}}) + 2h(D_{2^n} \times Z_4) + 2h(D_{2^{n-1}} \times C_8)$
$+ 4h(Z_{2^{n-2}} \times C_8) + 2^4 h(Z_{2^{n-3}} \times C_8) + 2^6 h(\mathbb{Z}_{2^{n-4}} \times C_8) - 2^8 h(\mathbb{Z}_{2^{n-5}} \times \mathbb{Z}_{2^3})$
$- 4h(\mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_{2^2}) + 2^{10}h(\mathbb{Z}_{2^{n-5}}) \times \mathbb{Z}_{2^2} - 2^9 h(\mathbb{Z}_{2^{n-5}}) - 2^9 h(D_{2^{n-4}} \times C_{2^2})$
$+ 2^8 h(D_{2^{n-4}} \times C_{2^3})$
$= 2^n + 2h(D_{2^n} \times C_4) + 2h(\mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_{2^3}) + 2^2 h(\mathbb{Z}_{2^{n-2}} \times \mathbb{Z}_{2^3})$
$- 2^{2(n-3)}h(\mathbb{Z}_{2^2} \times \mathbb{Z}_{2^3}) + 2^{2(n-2)}h(\mathbb{Z}_{2^2} \times \mathbb{Z}_{2^2} - 2^2 h(\mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_{2^2}) - 2^{2n-5}h(\mathbb{Z}_{2^2})$
$- 2^{2n-5}h(D_{2^3} \times \mathbb{Z}_{2^2}) + 2^{2(n-3)}h(D_{2^3} \times \mathbb{Z}_{2^3})$

$$+ \quad 3\sum_{i=1}^{n-5} 2^{2ij}h(\mathbb{Z}_{2^{n-2-i}} \times \mathbb{Z}_{2^3})$$

as required. $\square$

Theorem : Let $G = D_{2^4} \times \mathbb{C}_{2^4}$. **Then ,** $h(G) = 61384$

Proof : There exist seven maximal subgroups . Two isomorphic to $D_{2^4} \times \mathbb{C}_{2^3}$. **two isomorphic to** $D_{2^3} \times \mathbb{C}_{2^4}$. **two isomorphic to** $D_{2^4} \times \mathbb{C}_{2^2}$, **while the seventh is isomorphic to** $\mathbb{Z}_{2^4}$.

Hence , we have that : $\frac{1}{2}h(G) = 2h(D_{2^4} \times Z_{2^2}) + 2h(D_{2^4} \times Z_{2^3}) + 2h(D_{2^3} \times Z_{2^4}) - 6h(D_{2^3} \times \mathbb{Z}_{2^3}) - 6h(\mathbb{Z}_{2^4} \times \mathbb{Z}_{2^2}) - 3h(\mathbb{Z}_{2^3} \times \mathbb{Z}_{2^3}) - 6h(\mathbb{Z}_{2^4}) + 2h(D_{2^3} \times \mathbb{Z}_{2^3}) + 28h(\mathbb{Z}_{2^3} \times Z_{2^2}) + 2h(Z_{2^4} \times Z_{2^2}) + 2h(\mathbb{Z}_{2^4}) + h(Z_{2^3} \times \mathbb{Z}_{2^3}) - 35h(\mathbb{Z}_{2^3} \times \mathbb{Z}_{2^2}) + 21h(\mathbb{Z}_{2^3} \times \mathbb{Z}_{2^2}) - 7h(\mathbb{Z}_{2^3}\mathbb{Z}_{2^2}) + h(\mathbb{Z}_{2^3} \times \mathbb{Z}_{2^2})$
$= 2[h(D_{2^4} \times Z_{2^2}) + h(D_{2^4} \times Z_{2^3}) + h(D_{2^3} \times Z_{2^4}) - 2h(D_{2^3} \times \mathbb{Z}_{2^3}) - 2h(\mathbb{Z}_{2^4} \times \mathbb{Z}_{2^2}) - h(\mathbb{Z}_{2^3} \times \mathbb{Z}_{2^3}) + 4h(D_{2^3} \times \mathbb{Z}_{2^2}) - 3h(\mathbb{Z}_{2^4}) + \frac{1}{2}h(Z_{2^4})]$

$\therefore h(G) = 4[700 + 8416 + 10744 - 10752 \breve{} 1088 + 162 + 704 \breve{} 40]$
$= 4[15346] = 61384$ $\square$

3|Computation for $G = D_{2^4} \times \mathbb{C}_{2^n}, n \geq 4.$

Our computation on the algebraic fuzzy structure given actually has an outcome which involves multiple sums

Proof :
The maximal subgroups are :
$(D_{2^4} \times C_{2^{n-1}}), 2(D_{2^3} \times C_{2^n}), 2(D_{2^n} \times C_{2^2}), (D_{2^n} \times C_{2^3})$ **and** (C_{2^n}).
We have that : $\frac{1}{2}h(G) = h(D_{2^4} \times C_{n-1}) + 2h(D_{2^3} \times C_n) + 2h(D_{2^n} \times C_{2^2}) + h(D_{2^n} \times C_{2^3}) + h(C_{2^n}) - 6h(D_{2^3} \times \mathbb{Z}_{2^{n-1}}) - 6h(\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^2}) - 3h(\mathbb{Z}_{2^{n\breve{}1}} \times \mathbb{Z}_{2^3}) - 6h(\mathbb{Z}_{2^n}) + 2h(D_{2^3} \times C_{2^{n-1}}) + 28h(C_{2^{n-1}} \times C_{2^n}) + h(C_{2^{n-1}} \times C_{2^3}) + 2h(C_{2^n} \times C_{2^2}) + 2h(\mathbb{Z}_{2^n}) - 35h(C_{2^{n-1}} \times C_{2^2}) + 21h(C_{2^{n-1}} \times C_{2^2}) - 7h(C_{2^{n-1}} \times C_{2^2}) + h(C_{2^{n-1}} \times C_{2^2})$

$= h(D_{2^4} \times C_{2^{n-1}}) + 2h(D_{2^3} \times C_2^n) + 2h(D_{2^n} \times C_{2^2}) + h(D_{2^n} \times C_{2^3}) - 4h(D_{2^3} \times \mathbb{Z}_{2^{n-1}}) - 4h(\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^2}) - 2h(\mathbb{Z}_{2^{n\breve{}1}} \times \mathbb{Z}_{2^3}) + 8h(\mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_{2^2}) - 3h(\mathbb{Z}_{2^n})$

$$\frac{1}{2}h(G) = h(D_{2^4} \times \mathbb{Z}_{2^{n-k}}) + 2h(D_{2^3} \times \mathbb{Z}_{2^n}) - 4h(D_{2^3} \times \mathbb{Z}_{2^{n-k}}) - 4h(\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^2})$$

$$-2h(\mathbb{Z}_{2^{n-k}} \times \mathbb{Z}_{2^3}) + 8h(\mathbb{Z}_{2^{n-k}} \times \mathbb{Z}_{2^2}) + \sum_{j=1}^{k} h(D_{2^{n-1+j}} \times \mathbb{Z}_{2^3}) + 2\sum_{j=1}^{k} h(D_{2^{n-1+j}} \times \mathbb{Z}_{2^2}) - 3\sum_{j=1}^{k} h(\mathbb{Z}_{2^{n+1-j}})$$

$$-2\sum_{j=1}^{k-1} h(D_{2^3} \times \mathbb{Z}_{2^{n-j}}) + 4\sum_{j=1}^{k-1} h(D_{2^{n-j}} \times \mathbb{Z}_{2^2}) - 2\sum_{j=1}^{k-1} h(D_{2^n-j} \times \mathbb{Z}_{2^3}),$$

whence , $n - k = 4, \Rightarrow k = n - 4. \therefore h(G) = 2h(D_{2^4} \times \mathbb{Z}_{2^4}) + 4h(D_{2^3} \times \mathbb{Z}_{2^n}) - 8h(D_{2^3} \times \mathbb{Z}_{2^4}) - 8h(\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n}) - 4h(\mathbb{Z}_{2^4} \times \mathbb{Z}_{2^3}) + 16h(\mathbb{Z}_{2^4} \times \mathbb{Z}_{2^2}) +$

$$2\sum_{j=1}^{n-4} h(D_{2^{n-1+j}} \times \mathbb{Z}_{2^3}) + 4\sum_{j=1}^{n-4} h(D_{2^{n-1+j}} \times \mathbb{Z}_{2^2}) - 6\sum_{j=1}^{n-4} h(\mathbb{Z}_{2^{n+1-j}})$$

$$-4\sum_{j=1}^{n-5} h(D_{2^3} \times \mathbb{Z}_{2^{n-j}}) + 8\sum_{j=1}^{n-5} h(D_{2^n-j} \times \mathbb{Z}_{2^2}) - 4\sum_{j=1}^{n-5} h(D_{2^{n-j}} \times \mathbb{Z}_{2^3})$$

$$\therefore h(G) = 2^{n+3}(422 - n^2 - 5n) - 9n^2 + 356n - 29160 + 2\sum_{j=1}^{n-4} h(D_{2^{n-1+j}} \times \mathbb{Z}_{2^3})$$

$$+4\sum_{j=1}^{n-4} h(D_{2^{n-1+j}} \times \mathbb{Z}_{2^2}) - 6\sum_{j=1}^{n-4} h(\mathbb{Z}_{2^{n+1-j}}) - 4\sum_{j=1}^{n-5} h(D_{2^3} \times \mathbb{Z}_{2^{n-j}}) + 8\sum_{j=1}^{n-5} h(D_{2^n-j} \times \mathbb{Z}_{2^2}) - 4\sum_{j=1}^{n-5} h(D_{2^{n-j}} \times \mathbb{Z}_{2^3})$$

$$= 2^{n+3}(422 - n^2 - 5n) - 9n^2 + 356n - 29160 + \sum_{j=1}^{n-4}[2h(D_{2^{n-1+j}} \times \mathbb{Z}_{2^3}) + 4h(D_{2^{n-1+j}} \times \mathbb{Z}_{2^2}) - 6h(\mathbb{Z}_{2^{n+1-j}})]$$

$$-\sum_{j=1}^{n-5}[4h(D_{2^3} \times \mathbb{Z}_{2^{n-j}}) - 8h(D_{2^n-j} \times \mathbb{Z}_{2^2}) + 4h(D_{2^{n-j}} \times \mathbb{Z}_{2^3})]$$

Hence , proved as required □

4|Applications

The computations so far by the use of GAP (General AlgorithmAlgorithms and Programming) and the Inclusion - Exclusion Principle can be certified here as being very useful in the computations of the distinct number of fuzzy subgroups for the finite nilpotent p - groups .

5|INSTANCES

We have the following examples as parts surfacing from our computations so far. The readers may consider the examples below in tabular format.

Example 1 :

Table 1
Table Summarizing some Number of Distinct Fuzzy Subgroups of $(D_{2^3} \times C_{2^n})$ **FOR** ≥ 3

S/N for the Number of m	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$h(G) = (D_{2^3} \times C_{2^n}), n \geq 3$	**5376**	**10728**	**21506**	**43347**	**86536**	**173320**	**347098**	**694910**

Example 2 : Now, since the stipulated condition that $m \geq 3$ **must definitely be fulfilled then the readers may consider the examples below in tabular format.**

Table 2
Table Summarizing some Number of Distinct Fuzzy Subgroups of $(D_{2^4} \times C_{2^n})$ **FOR** $n \geq 4$

S/N for the Number of n	**4**	**5**	**6**
$h(G) = (D_{2^4} \times C_{2^n}), n \geq 4$	**20, 200**	**375, 648**	**3, 893, 800**

6|Conclusion

The discoveries from our studies so far , has helped to observe that any finite product of nilpotent group is nilpotent. Also, the problem of classifying the fuzzy subgroups of a finite group has experienced a very rapid progress. Finally, the method can be used in further computations up to the generalizations of similar and other given structures

Funding: **This research received no external funding**

Competing of interests statement : **The authors declare that in this paper, there is no competing of interests.**

Acknowledgments **The authors are grateful to the anonymous reviewers for their helpful comments and suggestions which has improved the overall quality and the earlier version of the work.**

References

[1] M. Mashinchi and M. Mukaidono (1992). A classification of fuzzy subgroups. Ninth Fuzzy System Symposium, Sapporo, Japan, 649-652.

[2] M. Mashinchi , M. Mukaidono .(1993). On fuzzy subgroups classification, Research Reports of Meiji Univ. (9) , 31-36.

[3] V. Murali and B. B. Makamba (2003). On an equivalence of Fuzzy Subgroups III, Int. J. Math. Sci. 36, 2303-2313.

[4] Odilo Ndiweni (2014). The classification of Fuzzy subgroups of the Dihedral Group D_n, for n, a product of distinct primes. A Ph.D. thesis, Univ. of Fort Hare, Alice, S.A.

[5] M. Tarnauceanu (2009). The number of fuzzy subgroups of finite cyclic groups and Delannoy numbers, European J. Combin. (30), 283-289, doi: 10.1016/j.ejc.2007.12.005.

[6] M. Tarnauceanu (2011). Classifying fuzzy subgroups for a class of finite p-groups. "ALL CUZa" Univ. Iasi, Romania.

[7] M. Tarnauceanu (2012). Classifying fuzzy subgroups of finite nonabelian groups. Iran.J.Fussy Systems. (9) 33-43

[8] M. Tarnauceanu , L. Bentea (2008). A note on the number of fuzzy subgroups of finite groups, Sci. An. Univ. "ALL.Cuza" Iasi, Matt., (54) 209-220.

[9] M. Tarnauceanu, L. Bentea (2008). On the number of fuzzy subgroups of finite abelian groups, Fuzzy Sets and Systems (159), 1084-1096, doi:10.1016/j.fss.2017.11.014

[10] S. A. ADEBISI and M. EniOluwafe (2020) The Abelian Subgroup : $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_{p^n}$, p is Prime and $n \geq 1$. Progress in Nonlinear Dynamics and Chaos Vol. 7, No. 1, 2019, 43-45 ISSN: 2321 – 9238 (online) Published on 21 September 2019 www.researchmathsci.orgDOI: http://dx.doi.org/10.22457/pindac.80v7n1a343

[11] S. A. ADEBISI, M. OGIUGO AND M. ENIOLUWAFE (2020) THE FUZZY SUBGROUPS FOR THE ABELIAN STRUCTURE: $h(\mathbb{Z}_8 \times \mathbb{Z}_{2^n})$ for $n > 2$. Journal of the Nigerian Mathematical Society, Vol. 39, Issue 2, pp. 167-171.

[12] S. A. ADEBISI, M. Ogiugo and M. EniOluwafe(2020)Computing the Number of Distinct Fuzzy Subgroups for the Nilpotent p-Group of $D_{2^n} \times C_4$ International J.Math.Combin.1(2020),86-89.

[13] S. A. ADEBISI, M. Ogiugo and M. EniOluwafe (2020) Determining The Number Of Distinct Fuzzy Subgroups For The Abelian Structure : $\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}$, $n > 2$.Transactions of the Nigerian Association of Mathematical Physics Volume 11, (January - June, 2020 Issue), pp 5 - 6.

[14] S. A. ADEBISI , M. OGIUGO and M. ENIOLUWAFE (2022)THE FUZZY SUBGROUPS FOR THE NILPOTENT (p-GROUP) OF $(D_{2^3} \times C_{2^m})$ FOR $m \geq 3$ Journal of Fuzzy Extension and Applications www.journal-fea.com (Accepted for publication).

[15] S. A. ADEBISI and M. EniOluwafe (2020) An explicit formula for the number of distinct fuzzy subgroups of the Cartesian product of the dihedral group of order 2n with a cyclic group of order 2 Universal Journal of Mathematics and Mathematical Sciences.http://www.pphmj.com http://dx.doi.org/10.17654/UM013010001 Volume 13, no1, 2020, Pages 1-7 ISSN: 2277-1417. (http://www.pphmj.com/journals/articles/1931.htm)

[16] S. A. ADEBISI, M. Ogiugo and M. EniOluwafe(2020) On the p-Groups of the Algebraic Structure of $D_{2^n} \times C_8$ International J.Math. Combin. Vol.3(2020), 100-103