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A New Weighted $T-X$ Perks Distribution: Characterization, Simulation And Applications

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Abstract

In this article, a new distribution is proposed to innovate the Perks distribution by altering its functional form without introducing additional parameter. The proposed distribution is named a new weighted T-X Perks (WT-XP) distribution. For this distribution, expressions for some mathematical properties are derived. The maximum likelihood estimates of the parameters α and β are derived and implemented for complete samples that follow the WT-XP distribution. To illustrate the importance of the proposed distribution over the other well-known distributions, two applications to real data sets are analyzed and the WT-XP distribution appear more attractive based on the Kolmogorov Smirnov statistic p-values and the model performance indicators used.

Keywords: Functional Form, Moment, Perks Distribution, Quantile Function, Weighted T-X Family.

1|Introduction

Probability distribution modelling of lifetime events has witnessed tremendous investigations in the last two decades. New distributions have been proposed within this time frame to fit data complexities better and

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improve the flexibility of applying probability distributions in today's world. Notable among events that have necessitated robust research in the probability field is the sudden advent of the Coronavirus 2019 (COVID-19 for short) and other numerous system failures in engineering and life sciences, which have spurred more work in reliability analysis.

Essentially, formulation of distributions is easier when there are generators such as the novel approach named the "Transformed-Transformer" otherwise called the $T-X$ generator credit to [57] or family of distributions. The $T-X$ generator has dominated the statistical literature since its deployment. This is due to the scarcity of robust methods that fit complex situations and enjoy parsimony. By parsimony, we mean that the $T-X$ family of distributions provides innovation for generating families of distributions without additional parameters. In other words, in most cases, the total number of parameters is the sum of the parameters of the "Transformer" distribution and the baseline (transformed) distribution. Again, the range of values of the random variable for the common generators before the $T-X$ family was proposed, namely the Beta-generated family by [**?**] and KW-G family by [1] are between 0 and 1, that is $x \in (0,1)$. This is a huge setback as numerous data encountered in research have their domains as the real line and hence are not bounded in a small interval such as $(0,1)$.

Due to the shortfalls of the existing generators that the $T-X$ family tried to overcome, it has gained wider applications. The following families of distributions have been proposed using the $T-X$ family of distribution generators. A new Flexible Logarithmic-X (NFLog-X) Family was proposed by [2]. Truncated family of distributions by [3]. [4] proposed the transmuted alpha power-G (TAP-G) family of distributions. [5] introduced the shifted exponential-G (SExpo-G) family of distributions. [6] proposed a new lifetime exponential-X family of distributions. [7] proposed the generalized alpha exponent power (GAEP) family of distributions. [8] proposed a new Generalization of the Gull Alpha Power (GGAP) family of distributions. [9] proposed a new generalized family of distributions based on combining Marshal-Olkin transformation with the $T-X$ family. The Frechet Topp Leone-G (FTL-G) family of distributions by [10]. [11] proposed a new modified exponent power alpha (NMEPA) family of distributions. [12] proposed the arcsine-X (Arcsine-X) family of distributions. [13] proposed the type-I heavy-tailed family of distributions. [14] proposed a new extended family of distributions. [15] developed the Gompertz-G family of distributions. For more flexible distributions with applications to lifetime data, see [16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40].

Ideally, any innovation aims to reduce complexities and solve problems. In the language of inference, it is called tractability and parsimony. On this basis, a few novel families of distributions are important to discuss. Exponentiated-G distributions, an interesting and apt form of extending a baseline distribution, were proposed by [41] with CDF given by

$$F(x;\alpha,\xi) = G(x;\alpha,\xi)^{\alpha}; \quad \alpha,\xi > 0, \quad x \in \Re \tag{1}$$

Marshall-Olkin-G family of distributions due to [42] is another great innovation in distribution theory. More improvement based on equation 1 are exponentiated generalized class of distributions by [43], exponentiated TX family of distributions by [44], Marshall-Olkin Chris-Jerry distribution by [45] and many others. Just as the Exponentiated distribution in equation 1, in the Marshall-Olkin-G family, only a parameter is added to a baseline distribution to innovate it. The cdf is given as

$$F(x;\alpha,\xi) = \frac{G(x;\xi)}{1-\bar{\alpha}\bar{G}(x;\xi)}; \quad \alpha,\xi > 0, \quad x \in \Re. \tag{2}$$

where $\bar{\alpha} = 1-\alpha;\quad G(x,\xi)$ is the CDF of the baseline distribution and $\bar{G}(x,\xi) = 1-G(x,\xi)$. Some improvement due to the Marshall-Olkin-G family includes the beta Marshall-Olkin family of distributions by [46], The Marshall–Olkin alpha power family of distributions by [47], The Marshall-Olkin generalized-G family of distributions by [48], Topp-Leone-Marshall-Olkin-G family of distributions by [49] and many more. [50] proposed one of the nicest families of distributions in the last few years, with CDF given by

$$F(x;\alpha,\xi) = \frac{e^{\alpha G(x;\xi)^2}-1}{e^{\alpha}-1}; \quad \alpha,\xi > 0, \quad x \in \Re. \tag{3}$$

where α is the additional parameter and $G(x,\xi)$ is the cdf of the baseline distribution. Equation 3 is named the Zubair-G family of distributions. The tractability of this family of distributions has necessitated more modifications such as Marshall-Olkin Zubair-G family of distributions by [51], Zubair Lomax distribution by [52], Zubair Gompertz distribution by [53], and Zubair-Exponential distribution by [54].

The Weighted T-X family introduced by [55] has cumulative distribution function (CDF) and probability density function (PDF) given as

$$G(x) = 1 - \left(\frac{1 - F(x;\Xi)}{e^{F(x;\Xi)}} \right), \tag{4}$$

and

$$g(x) = \frac{f(x;\Xi)}{e^{F(x;\Xi)}} \left\{ 2 - F(x;\Xi) \right\}, \tag{5}$$

respectively with Ξ being the vector of parameters of any baseline distribution having its CDF and PDF are $F(x;\Xi)$ and $f(x;\Xi)$. The interesting feature of this family of distributions is that it does not introduce additional parameter(s) to the baseline distribution, rather alters the functional form of the baseline. The Perks distribution by [56] has CDF and PDF given as

$$F(x;\alpha,\beta) = 1 - \left(\frac{1+\alpha}{1+\alpha e^{\beta x}} \right); \quad x > 0, \quad \alpha, \beta > 0 \tag{6}$$

and

$$f(x;\alpha,\beta) = \frac{(1+\alpha)\alpha\beta e^{\beta x}}{(1+\alpha e^{\beta x})^2} \tag{7}$$

Inserting equation 6 and 7 into 4 and 5, the Weighted T-X Perks (WT-XP) distribution is obtained with CDF and PDF given as

$$G(x;\alpha,\beta) = 1 - \frac{1+\alpha}{(1+\alpha e^{\beta x})\, e^{1 - \frac{1+\alpha}{1+\alpha e^{\beta x}}}}; \quad x > 0, \quad \alpha, \beta > 0, \tag{8}$$

and

$$g(x;\alpha,\beta) = \frac{\alpha(\alpha+1)\beta\left(\alpha\left(\mathrm{e}^{\beta x}+1\right)+2\right)\mathrm{e}^{\frac{\alpha+1}{\alpha \mathrm{e}^{\beta x}+1}+\beta x - 1}}{\left(\alpha \mathrm{e}^{\beta x}+1\right)^3}, \tag{9}$$

respectively.

The hazard function is

$$h(x;\alpha,\beta) = \frac{(1+\alpha)\alpha\beta e^{\beta x}\left[2+\alpha+\alpha e^{\beta x}\right]}{(1+\alpha e^{\beta x})(1+\alpha)e^{\frac{1+\alpha}{1+\alpha e^{\beta x}}}}. \tag{10}$$

The graphs of the PDF in figure 1 exhibit varying shapes making the distribution attractive for modeling lifetime data that exhibit similar characteristics. The hazard functions in figures 3 and 4 are uniquely left-skewed, obviously showing scenario of increasing failure rate. This situation abound in real-life. Example include the mortality rate of patients diagnosed of certain diseases/epidemics from the outbreak to a certain peak.

1.1|Linear Representation of the PDF of WT-XP (α, β) Distribution

To make equation 9 tractable, we deploy the binomial expansion technique which leads to

$$g(x;\alpha,\beta) = (\alpha+1)\alpha\beta\Phi_{i,j,k,l,m,n}\left\{\alpha x^{j-k-n}e^{(i+l+m)\beta x} + (\alpha+2)x^{j-k-n}e^{(i+m)\beta x}\right\}, \tag{11}$$

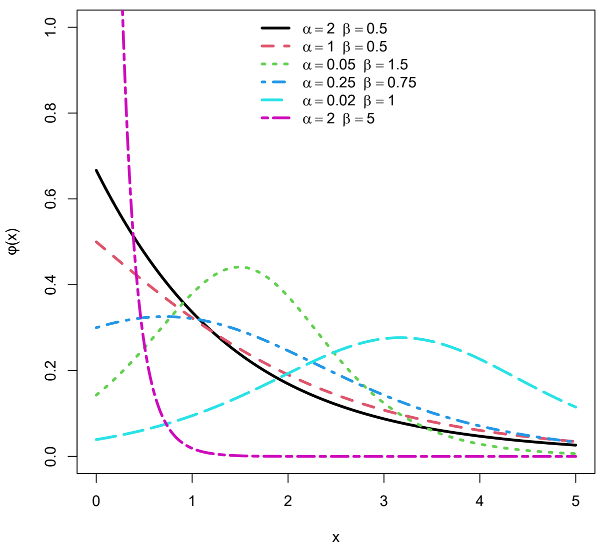


FIGURE 1. PDF of WT-XP (α, β)

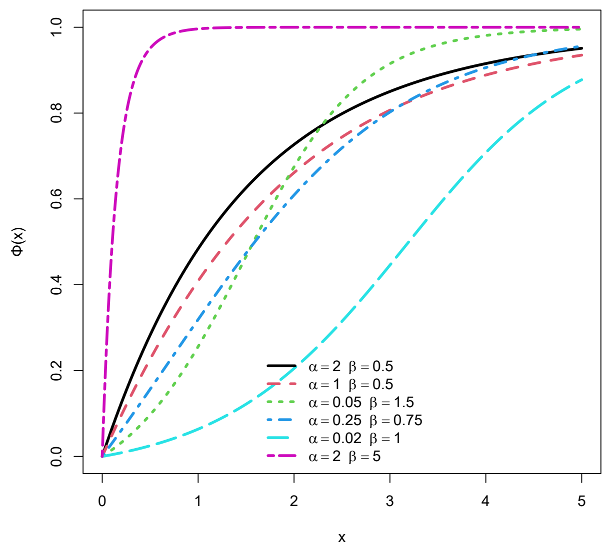


FIGURE 2. CDF of WT-XP (α, β)

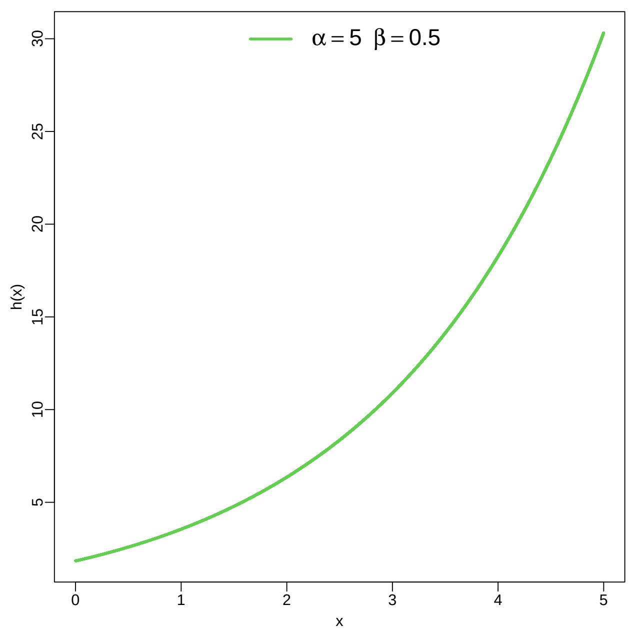


FIGURE 3. HRF of WT-XP (α, β)

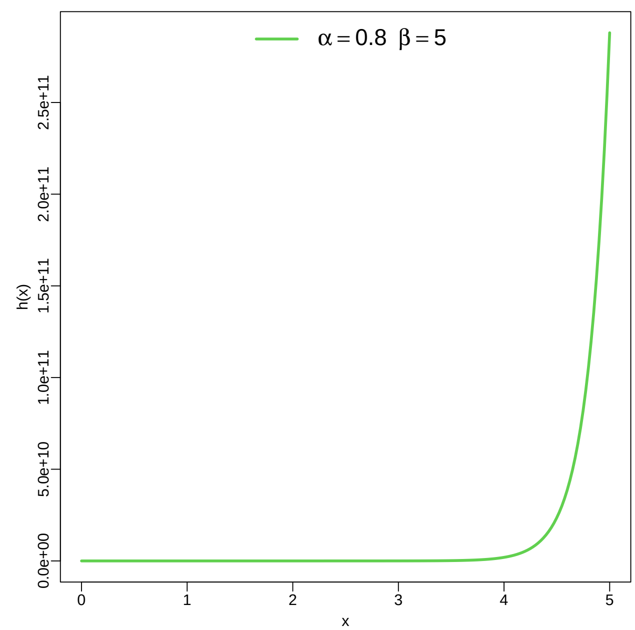


FIGURE 4. HRF of WT-XP (α, β)

where

$$\Phi_{i,j,k,l,m,n} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{j-k} \frac{(-1)^{i+m+j-k}}{j!}\binom{2+i}{i}\binom{j}{k}\binom{k}{l}\binom{k+m-1}{m}\binom{j-k}{n}\alpha^{i+l+m}\beta^{j-k-n}.$$

The remaining sections of this article are in the following arrangement; section discusses some properties of the distribution that are analytically tractable which includes the crude moment, the moment generating function and the quantile function. In section , we estimate the parameters of the distribution by means of the maximum likelihood estimation. Section presents the behaviour of the model through simulation study. In section we consider two real life applications and concluded the article in section .

2|Mathematical Characteristics of the WT-XP (α, β) Distribution

In this section, we study the basic properties of the distribution. For clarity, the characteristics studied are the tractable ones namely, the moment and related indices, the moment generating functions and the quantile function.

2.1|The rth Crude Moment

Given $X \sim$ WT-XP (α, β), the rth non-central moment is given by

$$\mu_r' = (\alpha+1)\alpha\beta\Phi_{i,j,k,l,m,n}\Gamma(j-k-n+r+1)\left\{\alpha\left(-\frac{1}{(i+l+m)\beta}\right)^{j-k-n+r+1} + (\alpha+2)\left(-\frac{1}{(i+m)\beta}\right)^{j-k-n+r+1}\right\}$$
for $\quad r = 1, 2, \cdots$
(12)

where

$$\Phi_{i,j,k,l,m,n} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{j-k}\frac{(-1)^{i+m+j-k}}{j!}\binom{2+i}{i}\binom{j}{k}\binom{k}{l}\binom{k+m-1}{m}\binom{j-k}{n}\alpha^{i+l+m}\beta^{j-k-n}.$$

Proof: From definition, the rth crude or non-central moment is given by $\mathrm{E}(X^r) = \mu_r' = \int_0^\infty x^r g(x;\alpha,\beta)\,dx$.

$$\mu_r' = (\alpha+1)\alpha\beta\Phi_{i,j,k,l,m,n}\left\{\alpha\int_0^\infty x^{j-k-n+r}e^{(i+l+m)\beta x}\,dx + (\alpha+2)\int_0^\infty x^{j-k-n+r}e^{(i+m)\beta x}\,dx\right\} \tag{13}$$

Define $-z = (i+l+m)\beta x$ and $-f = (i+m)\beta x$, then

$$\mu_r' = (\alpha+1)\alpha\beta\Phi_{i,j,k,l,m,n}\Gamma(j-k-n+r+1)\left\{\alpha\left(-\frac{1}{(i+l+m)\beta}\right)^{j-k-n+r+1} + (\alpha+2)\left(-\frac{1}{(i+m)\beta}\right)^{j-k-n+r+1}\right\}. \tag{14}$$
□

The mean of WT-XP (α, β) is attained when $r = 1$ in equation 14.

$$\mu = (\alpha+1)\alpha\beta\Phi_{i,j,k,l,m,n}\Gamma(j-k-n+2)\left\{\alpha\left(-\frac{1}{(i+l+m)\beta}\right)^{j-k-n+2} + (\alpha+2)\left(-\frac{1}{(i+m)\beta}\right)^{j-k-n+2}\right\}. \tag{15}$$

Similarly, the second, third and fourth crude moments are attained when $r = 2, 3$ and 4 in equation 14.

$$\mu_2' = (\alpha+1)\alpha\beta\Phi_{i,j,k,l,m,n}\Gamma(j-k-n+3)\left\{\alpha\left(-\frac{1}{(i+l+m)\beta}\right)^{j-k-n+3} + (\alpha+2)\left(-\frac{1}{(i+m)\beta}\right)^{j-k-n+3}\right\}, \tag{16}$$

$$\mu_3' = (\alpha+1)\alpha\beta\Phi_{i,j,k,l,m,n}\Gamma(j-k-n+4)\left\{\alpha\left(-\frac{1}{(i+l+m)\beta}\right)^{j-k-n+4} + (\alpha+2)\left(-\frac{1}{(i+m)\beta}\right)^{j-k-n+4}\right\}, \tag{17}$$

and

$$\mu_4' = (\alpha+1)\alpha\beta\Phi_{i,j,k,l,m,n}\Gamma(j-k-n+5)\left\{\alpha\left(-\frac{1}{(i+l+m)\beta}\right)^{j-k-n+5} + (\alpha+2)\left(-\frac{1}{(i+m)\beta}\right)^{j-k-n+5}\right\}. \tag{18}$$

2.2|Moment Generating Function

Moments generally referred to as the mean value of the powers of a random variable say X, where they exists and are finite can be obtained using a generating function called the moment generating function $M-X(t) = \mathrm{E}(e^{tx})$ defined for all positive $t \in \Re$. So,

$$\begin{aligned} M_X(t) &= (\alpha+1)\alpha\beta\Phi_{i,j,k,l,m,n}\left\{\alpha\int_0^\infty x^{j-k-n}e^{[(i+l+m)\beta+t]x}\,dx + (\alpha+2)\int_0^\infty x^{j-k-n}e^{[(i+m)\beta+t]x}\,dx\right\} \\ &= (\alpha+1)\alpha\beta\Phi_{i,j,k,l,m,n}\Gamma(j-k-n+1)\left\{\alpha\left(-\frac{1}{(i+l+m)\beta+t}\right)^{j-k-n+1} + (\alpha+2)\left(-\frac{1}{(i+m)\beta+t}\right)^{j-k-n+1}\right\} \end{aligned} \tag{19}$$

where

$$\Phi_{i,j,k,l,m,n} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{j-k}\frac{(-1)^{i+m+j-k}}{j!}\binom{2+i}{i}\binom{j}{k}\binom{k}{l}\binom{k+m-1}{m}\binom{j-k}{n}\alpha^{i+l+m}\beta^{j-k-n}.$$

2.3|Quantile Function

The quantile when it is analytically plausible provides basis for data generation. This aids in the simulation of random sample for the determination of model behaviour. Here, the CDF in equation 8 is inverted by first writing $G(x;\alpha,\beta) = q$, so that

$$\begin{aligned} 1-q &= \frac{1+\alpha}{\left(1+\alpha e^{\beta x}\right)e^{1-\frac{1+\alpha}{1+\alpha e^{\beta x}}}} \\ \implies \left(1+\alpha e^{\beta x}\right)e^{\left(1+\alpha e^{\beta x}\right)^{-1}} &= \frac{(1+\alpha)\,e^{(2+\alpha)}}{1-q} \end{aligned} \tag{20}$$

Let $\left(1+\alpha e^{\beta x}\right)^{-1} = Z(x)$; so that $-Z(x)e^{-Z(x)} = -\frac{(1-q)e^{-(2+\alpha)}}{1+\alpha}$. Taking Lambert W function ([58, 59]), we obtain

$$\begin{aligned} -Z(x) &= W\left[-\frac{(1-q)e^{-(2+\alpha)}}{1+\alpha}\right] \\ -\frac{1}{1+\alpha e^{\beta x}} &= W\left[-\frac{(1-q)e^{-(2+\alpha)}}{1+\alpha}\right] \end{aligned} \tag{21}$$

Hence, the quantile function is

$$x_q = \frac{1}{\beta}\ln\left[\frac{1}{\alpha}\left\{-1-\frac{1}{W\left[-\frac{(1-q)e^{-(2+\alpha)}}{1+\alpha}\right]}\right\}\right]. \tag{22}$$

The median lifetime of X is attained when $q = \frac{1}{2}$ in equation 22.

$$x_{\frac{1}{2}} = \frac{1}{\beta} \ln \left[\frac{1}{\alpha} \left\{ -1 - \frac{1}{W\left[-\frac{e^{-(2+\alpha)}}{2(1+\alpha)} \right]} \right\} \right]. \tag{23}$$

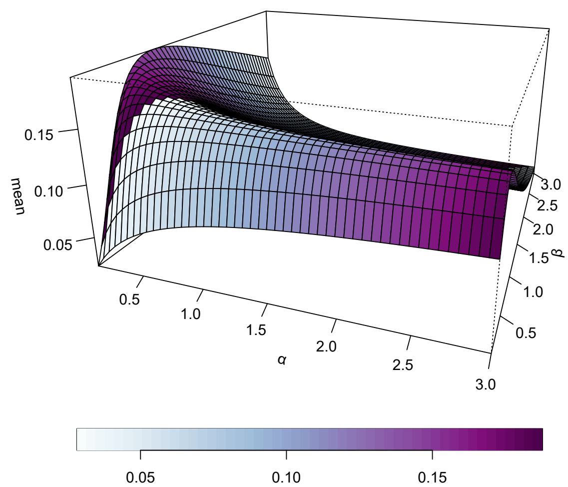


FIGURE 5. Mean of WT-XP (α, β)

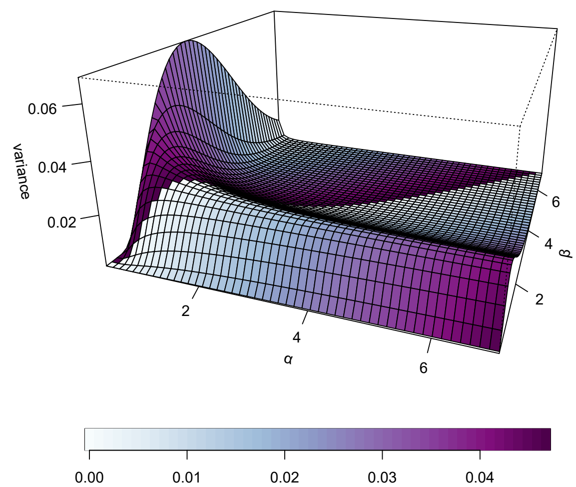


FIGURE 6. Variance of WT-XP (α, β)

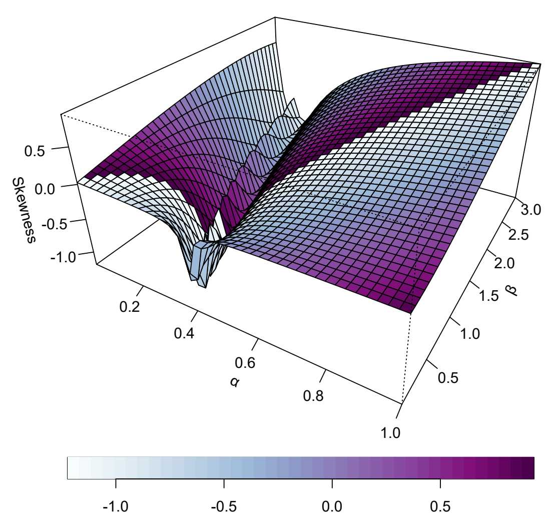


FIGURE 7. Skewness of WT-XP (α, β)

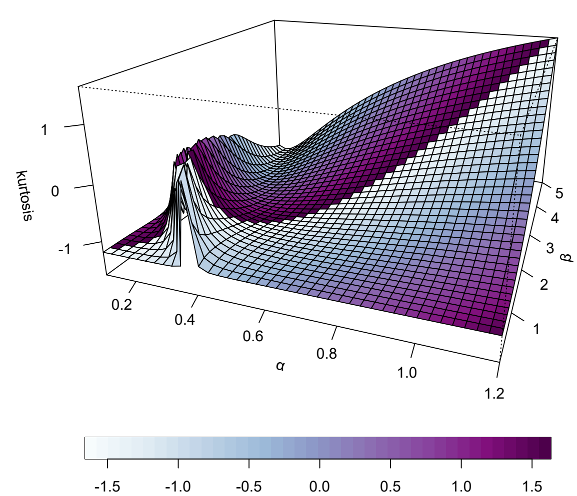


FIGURE 8. Kurtosis of WT-XP (α, β)

Figures 5, 6, 7 and 8 are the plots of the mean, variance, skewness and kurtosis.

3|Point Estimation of Parameters

The estimation of parameters is one the important aspects of modeling. In this section, the maximum likelihood is used to find the estimates of α and β. Suppose $X_{(1)}, X_{(2)}, \cdots, X_{(k)}$ are ordered independent samples of sizes n each following the proposed WT-XP (α, β) with a joint PDF $g(x_{(i)}; \alpha, \beta)$, the likelihood function $L(X\big|\alpha, \beta)$ is written as;

$$L(X\big|\alpha,\beta) = (1+\alpha)^n\,\alpha^n\beta^n e^{\beta\sum\limits_{i=1}^{n}x_i}\, e^{-n+\sum\limits_{i=1}^{n}\left(\frac{1+\alpha}{1+\alpha e^{\beta x_i}}\right)} \prod_{i=1}^{n}\left(1+\alpha e^{\beta x}\right)^{-2}\left[\frac{2+\alpha\left(1+e^{\beta x}\right)}{1+\alpha e^{\beta x}}\right] \tag{24}$$

Take the log of $L(.)$ and assign it to ℓ

$$\begin{aligned}\ell &= n\log(1+\alpha) + n\log\alpha + n\log\beta + \beta\sum_{i=1}^{n}x_i - n + \sum_{i=1}^{n}\left(\frac{1+\alpha}{1+\alpha e^{\beta x}}\right)\\ &- 2\sum_{i=1}^{n}\log\left(1+\alpha e^{\beta x}\right) + \sum_{i=1}^{n}\log\left[2+\alpha\left(1+e^{\beta x}\right)\right] - \sum_{i=1}^{n}\log\left(1+\alpha e^{\beta x}\right)\end{aligned} \tag{25}$$

The first-order derivatives of ℓ with respect to α and β are respectively;

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{1+\alpha} + \frac{n}{\alpha} - 3\sum_{i=1}^{n}\frac{1}{\alpha+e^{-\beta x}} + \sum_{i=1}^{n}\frac{1}{\alpha+2\left(1+e^{\beta x}\right)^{-1}} \tag{26}$$

and

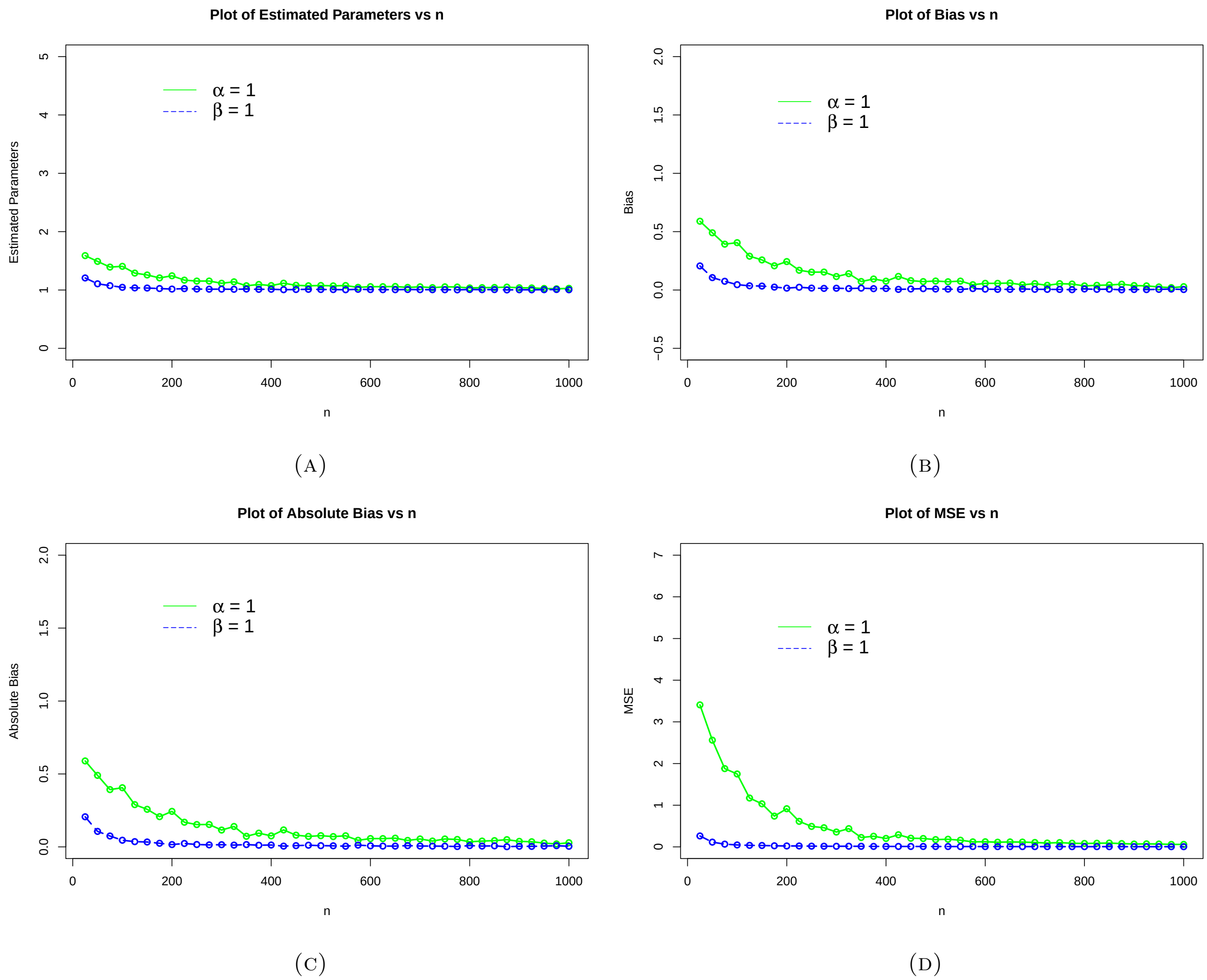
$$\frac{\partial\ell}{\partial\beta} = \frac{n}{\beta} + \sum_{i=1}^{n}x_i - \alpha(1+\alpha)\sum_{i=1}^{n}\frac{xe^{\beta x}}{\left(1+\alpha e^{\beta x}\right)^2} - 2\alpha\sum_{i=1}^{n}\frac{x}{\alpha+e^{-\beta x}} + \alpha\sum_{i=1}^{n}\left[\frac{x}{\alpha+(2+\alpha)e^{-\beta x}} - \frac{x}{\alpha+e^{-\beta x}}\right] \tag{27}$$

The non-linear system of equations 26 and 27 do not possess analytical solution. Numerical iteration using Newton-Raphson's algorithm is used to facilitate the convergence of $\hat{\alpha}$ and $\hat{\beta}$. In R, a special function called **optim()** function ([60]) is employed for this optimization.

4|Behaviour of the WT-XP (α, β) Model via Simulation

In this section, simulation study is conducted to ascertain the behaviour of WT-XP (α, β) distribution parameters as sample size n increases. Sample sizes $n = 25, 50, 75, \cdots, 1000$ with 1000 replications in each case were chosen and the average estimates of the parameters, bias, absolute bias, and the mean squared error are obtained. Initial parameter values of $(\alpha = 1, \beta = 1)$ and $(\alpha = 0.25, \beta = 1.5)$ are the two scenarios for the simulation whose graphical illustrations are presented in figures 9 and 10. From both figures, we see that α decays faster as the sample size becomes large. Both parameters tends to zero as the sample size increases. The bias is positive and also reduces as the sample size becomes large. This is a good asymptotic result and shows that distribution can be beneficial in modeling lifetime events. The computational formulae for the bias, absolute bias and mean square error are respectively;

$$\begin{aligned}\text{Bias}(\Xi) &= \frac{1}{N}\sum_{i=1}^{N}(\hat{\Xi}_i - \Xi),\\ \text{ABias}(\Xi) &= \frac{1}{N}\sum_{i=1}^{N}\mid\hat{\Xi}_i - \Xi\mid,\\ \text{MSE}(\Xi) &= \frac{1}{N}\sum_{i=1}^{N}(\hat{\Xi}_i - \Xi)^2,\end{aligned} \tag{28}$$



(A) (B) (C) (D)

FIGURE 9. Simulation Results for WT-XP ($\alpha = 1, \beta = 1$)

where Ξ is the vector of parameters of the suggested WT-XP distribution. Hence we write $\Xi = (\alpha, \beta)$ which is estimated by $\hat{\Xi}$.

5|Applications

This section demonstrates the utility of the proposed WT-XP distribution based on two real data sets. Data-I consists of volume of Bitcoin in USD (BTC-USD) traded weekly from July 17, 2023 to July 19, 2024, as mentioned in table 1. The data can be found https://finance.yahoo.com/quote/BTC-USD/history/ (accessed on 19 July 2024). The second data represent the infant mortality rate of China from 1969 to 2021. The data are contained in table 2 and obtained from https://data.worldbank.org/indicator/SP.DYN.IMRT.IN (accessed on 20 July 2024).

The competing models chosen are

(1) The **Perks** distribution with CDF

$$F(x;\alpha,\beta) = 1 - \left(\frac{1+\alpha}{1+\alpha e^{\beta x}}\right); \quad x > 0, \quad \alpha, \beta > 0 \quad \text{and PDF} \quad f(x;\alpha,\beta) = \frac{(1+\alpha)\alpha\beta e^{\beta x}}{(1+\alpha e^{\beta x})^2}.$$

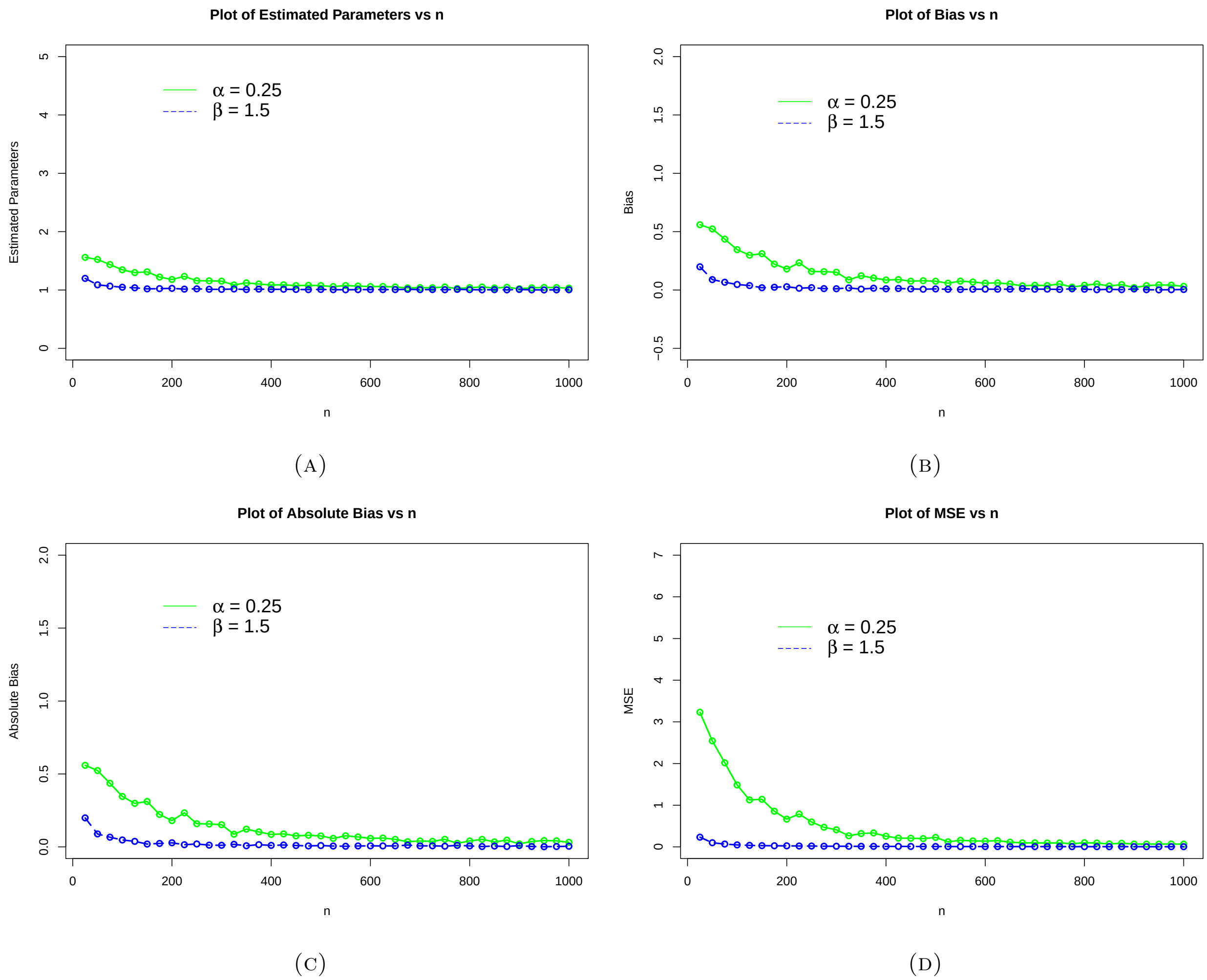


FIGURE 10. Simulation Results for WT-XP $(\alpha = 0.25, \beta = 1.5)$

(2) The **Weibull** distribution with PDF is given by:

$$f(x;\alpha,\beta) = \beta\alpha x^{\alpha-1}e^{-\beta x^{\alpha}} \quad \text{and CDF} \quad F(x;\alpha,\beta) = 1 - e^{-\beta x^{\alpha}}.$$

(3) **Gamma** distribution with PDF

$$f(x;\alpha,\beta) = \frac{x^{\alpha-1}e^{-\beta x}\beta^{\alpha}}{\Gamma(\alpha)} \quad \text{for } x > 0 \text{ and } \alpha, \beta > 0, \quad \text{and CDF} \quad F(x;\alpha,\beta) = \frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}.$$

(4) **Gumbel** distribution with CDF

$$F(x;\mu,\beta) = e^{-e^{-(x-\mu)/\beta}}; \quad \mu > 0, \beta > 0 \quad \text{and PDF} \quad f(x;\mu,\beta) = e^{-(x-\mu)/\beta - e^{-(x-\mu)/\beta}}.$$

(5) The **log-normal** distribution with PDF

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right); \quad \mu > 0, \sigma > 0 \quad \text{and CDF} \quad F(x) = \Phi\left(\frac{\ln x - \mu}{\sigma}\right).$$

For model performance, the log-likelihood statistic (LL), the akaike information criterion (AIC), the corrected akaike information criterion (CAIC), the bayesian information criterion (BIC), hannan quinn information criterion (HQIC) are the metrics. The decision is such that the distribution the least values of these criteria is considered best. For goodness of fit, the cramér von misses statistic (W), the anderson-darling statistic (A), the kolmogorov-smirnov statistic (KS) and its p-value are considered. With a p-value of 0.9254 which is higher than those of the competing distributions, the suggested WT-XP distribution fits the BTC-USD data better than the rest of the models. The same is applicable to the China infant mortality rate data which has a p-value of 0.8025. Figures 11 and 12 are evidences of how well the WT-XP distribution fits the two data sets respectively.

Table 1. Bitcoin in USD (BTC-USD) volume traded weekly from July 17, 2023 to July 19, 2024 (Data-I)

5.48500454	7.63096295	8.782700105	8.515301445	11.6418275	8.308680745	11.31616278
6.981461115	8.579669986	8.225190521	7.301224073	8.21837757	7.267365449	11.34023831
16.5718773	11.55200566	13.47674948	14.35353611	13.0439894	13.4360108	18.85036149
16.8493482	15.20841715	15.78071262	20.2311774	25.99718197	13.91358248	15.53146029
13.51841874	15.78263954	21.35600415	16.07691456	32.28956661	40.57096933	40.59577506
32.80971217	21.63246258	22.94272601	28.82436754	25.78895559	17.60310882	21.77170118
16.16178571	19.1456349	22.5779718	17.09494099	18.75695007	17.59732716	16.34490154
17.10300939	21.31075314	18.740702	13.94762495	3.348603085		

Table 2. Infant mortality rate of China from 1969 to 2021

118.8	112.8	107.1	101.4	95.9	90.3	84.8	79.5	74.5	70.1	66.1	62.7	59.9	57.7	56
54.8	54.1	53.8	53.8	53.9	53.9	53.6	53.1	52.1	50.8	49.2	47.4	45.5	43.5	41.4
39.1	36.7	34.1	31.5	28.9	26.3	24	22	20.1	18.5	17.1	15.8	14.6	13.5	12.5
11.6	10.7	9.9	9.2	8.6	8	7.4	6.9							

Table 3. Model Comparison and Fitness Measures for USD (BTC-USD) Data

Dist	LL	AIC	CAIC	BIC	HQIC	W	A	K-S	p-value	$\hat{\alpha}_{\text{MLE}}$	$\hat{\beta}_{\text{MLE}}$
WT-XP	-185.31	374.623	374.858	378.601	376.157	0.071	0.543	0.072	0.9254	0.0193	0.2047
Gamma	-183.49	370.971	371.206	374.949	372.505	0.065	0.408	0.078	0.8690	4.5227	0.2719
Weibull	-185.17	374.344	374.579	378.322	375.878	0.091	0.634	0.095	0.6818	18.8229	2.2164
Gumbel	-183.49	370.987	371.223	374.965	372.521	0.062	0.384	0.087	0.7737	13.0591	6.1599
LNORM	-184.19	372.385	372.621	376.363	373.920	0.105	0.562	0.106	0.5431	2.6967	0.4943

Table 4. Model Comparison and Fitness Measures for Infant Mortality Rate of China

Dist	LL	AIC	CAIC	BIC	HQIC	W	A	KS	p-value	$\hat{\alpha}_{\text{MLE}}$	$\hat{\beta}_{\text{MLE}}$
WT-XP	-249.69	503.374	503.614	507.314	504.889	0.112	0.710	0.0883	0.8025	0.1076	0.0419
Perks	-249.62	503.239	503.479	507.180	504.754	0.112	0.705	0.0893	0.7919	0.2204	0.0452
Gamma	-249.25	502.509	502.749	506.449	504.024	0.162	0.894	0.1233	0.3959	2.0327	0.0444
Weibull	-248.76	501.514	501.754	505.455	503.029	0.125	0.722	0.1021	0.6391	50.9903	1.5706
Gumbel	-251.69	507.371	507.611	511.312	508.887	0.141	0.866	0.1092	0.5523	31.9212	23.6552
LNORM	-251.69	507.384	507.624	511.324	508.899	0.277	1.500	0.1567	0.1480	3.5577	0.7962

6|Conclusion

This article modifies the functional form of the popular Perks distribution. The modification produced a better distribution without introducing an additional parameter. The plots of the PDF, CDF and the hazard function illustrate the flexibility of the model since its possesses varying interesting shapes. For the record, the new distribution has some tractable properties which were derived in the article namely the crude moment, moment generating function, and the quantile function. The model usefulness was demonstrated using the bitcoin weekly

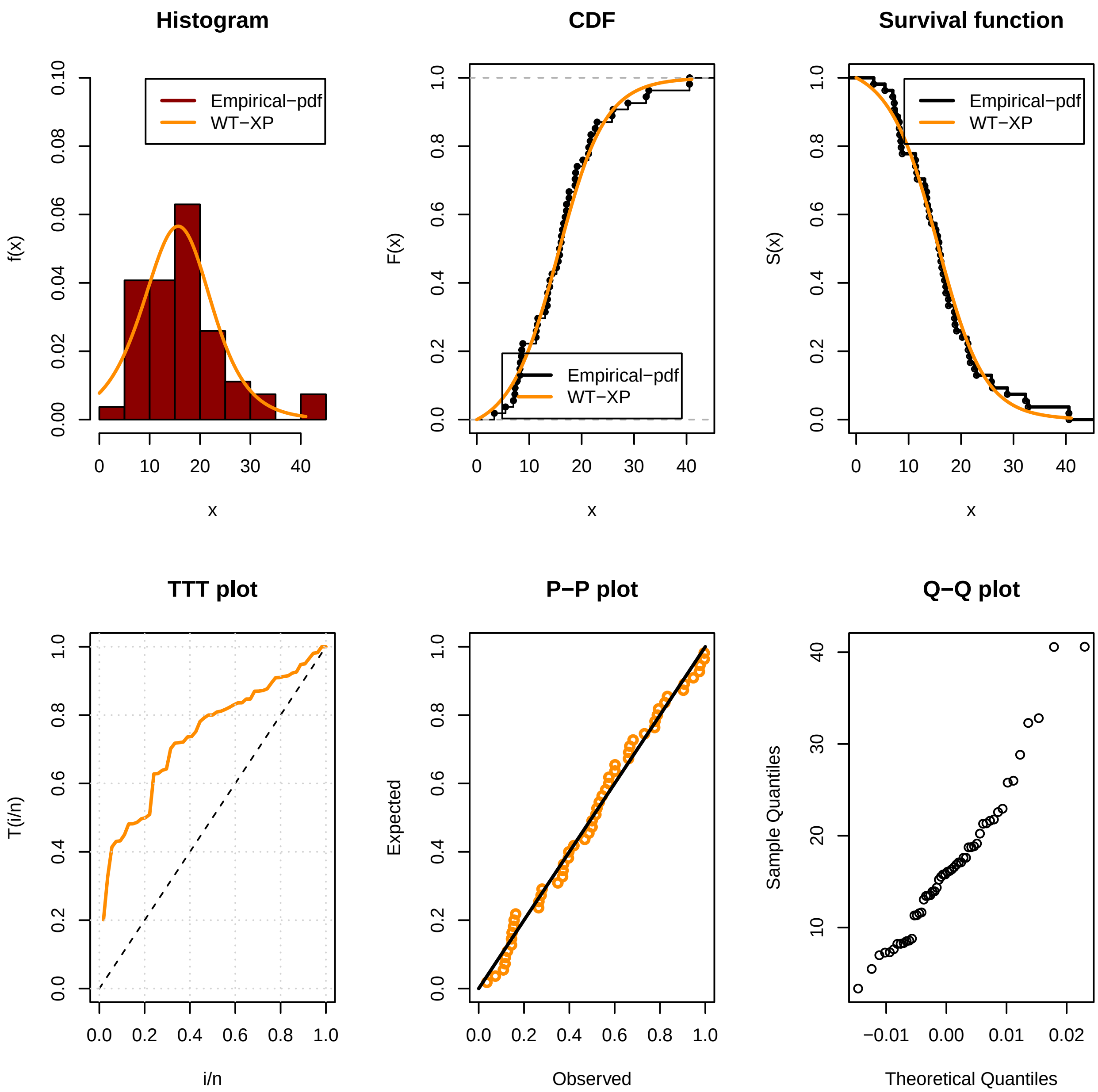


FIGURE 11. Plots of the USD (BTC-USD) Data

volume of trade and the China infant mortality rate data. Financial and health records are gray areas of human engagements. The ability of this distribution to fit data from these two vital sectors of human life portrays this distribution ad one that fills a huge gap in the literature. Similarly, the fact that it beats the classical distributions namely the Weibull, Gamma, Gumbel and log-normal even the baseline Perks distribution is sufficient to say that the distribution is very appealing.

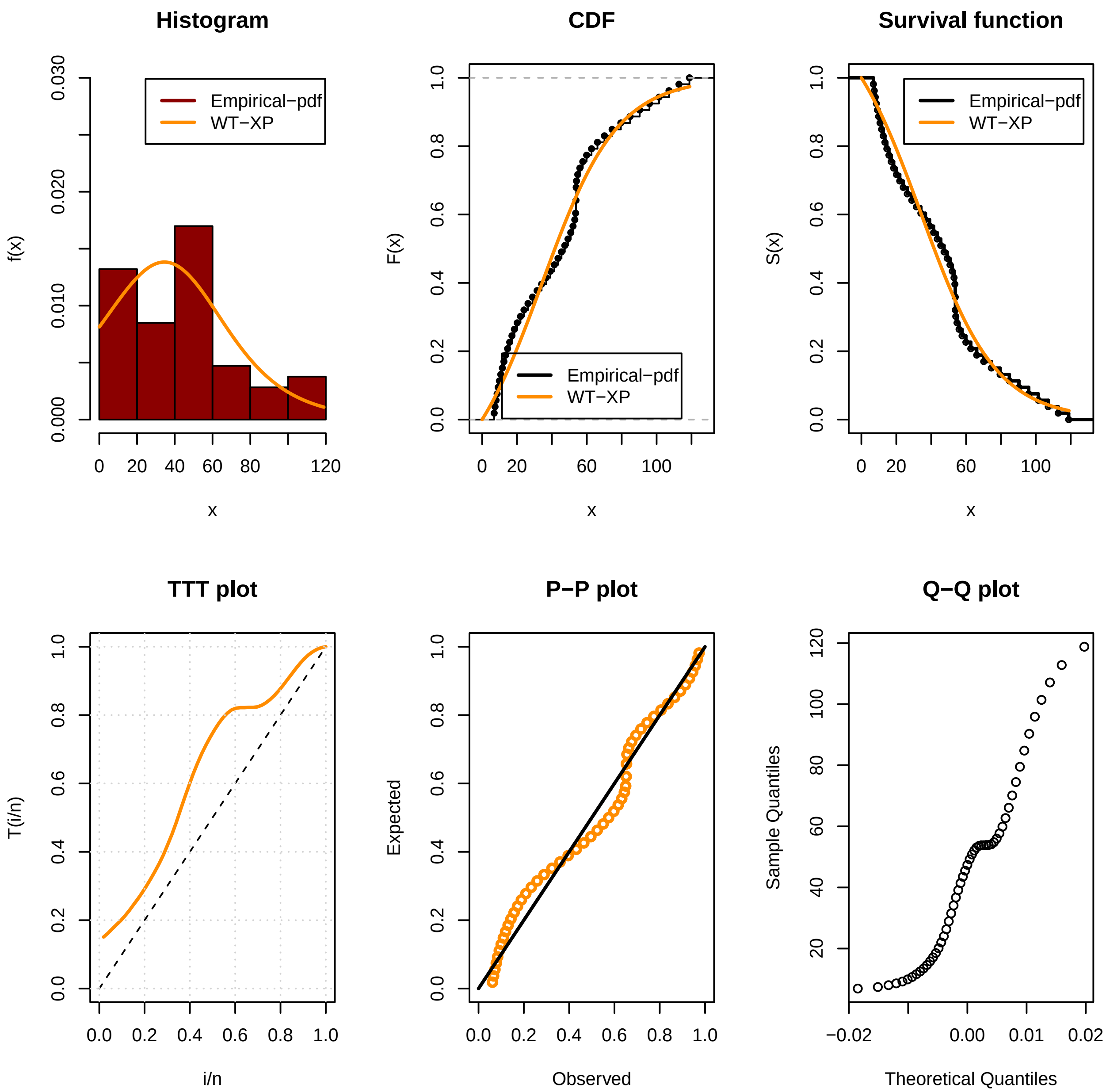


Figure 12. Plots of the Infant Mortality Rate of China

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Author Contribution

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Conflicts of Interest

The authors declare that there is no conflict of interest concerning the reported research findings. Funders played no role in the study's design, in the collection, analysis, or interpretation of the data, in the writing of the manuscript, or in the decision to publish the results.

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